Unstable Particle in Quantum Mechanics

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Exercise:

We consider a simple quantum mechanical model of an unstable particle which decays with some lifetime τ.

Solution:

Suppose $P(t) = \int_{-\infty}^{\infty} |\Psi|^2 dx$. And suppose Ψ is a solution of the Schrödinger equation with potential $V = V_0 - i\Gamma,$ where $\Gamma \in \mathbb{R}_{>0}$. Then,

$$\frac{dP(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty}\Big(\dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}\Big)dx =$$

$$\frac{i}{\hbar}\int_{-\infty}^{\infty}\Big[\Big(-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi + (V_0 - i\Gamma)|\Psi|^2\Big) - \Big(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xx} + (V_0 + i\Gamma)|\Psi|^2\Big)\Big]dx =$$

$$-\frac{2\Gamma}{\hbar}P(t) + \frac{i}{\hbar}\int_{-\infty}^{\infty} -\frac{\hbar^2}{2m}\Big(\Psi^*_{xx}\Psi - \Psi^*\Psi_{xx}\Big)dx = -\frac{2\Gamma}{\hbar}P(t) \implies$$

$$\frac{dP(t)}{dt} = -\frac{2\Gamma}{\hbar}P(t).$$

Solving this equation, we get

$$\boxed{P(t) = e^{-2\Gamma t/\hbar} \equiv e^{-t/\tau}, \text{ where } \tau = \hbar/2\Gamma.}$$

Thus, we conclude that a solution to the Schrödinger equation with a real potential and a constant imaginary offset describes an unstable particle.