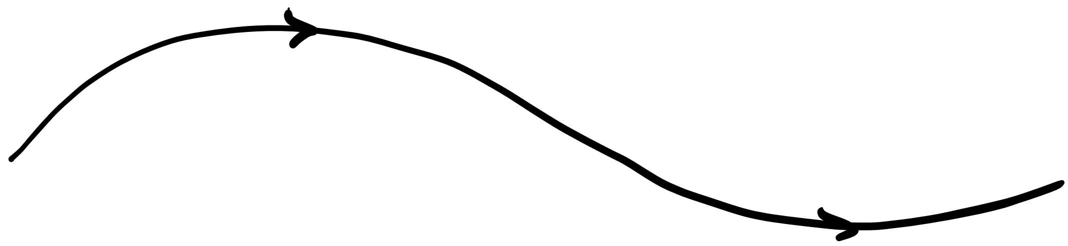
Exercise: We derive the translation operator $\hat{T}(a)$ in quantum mechanics.

(Source: Griffiths QM Section 6.2)



We wish to find an operator $\hat{T}(a)$ such that $\hat{T}(a)\Psi(x) = \Psi(x-a)$.

We first recall the Taylor expansion of $\Psi(y)$ near $y = x$.

$$\Psi(y) = \frac{\Psi(x)}{0!} + \frac{\Psi'(x)}{1!}(y-x) + \frac{\Psi''(x)}{2!}(y-x)^2 + \cdots + \frac{\Psi^{(n)}}{n!}(y-x)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(y-x)^n \frac{d^n}{dx^n}\Psi(x) = e^{(y-x)\frac{d}{dx}}\Psi(x)$$

Now, let $y = x - a$. Then, we have

$$\Psi(x-a) = e^{-a\frac{d}{dx}}\Psi(x) = e^{-(ia/\hbar)(-i\hbar\frac{d}{dx})}\Psi(x)$$

$$= e^{-ia\hat{p}/\hbar}\Psi(x)$$, where $\hat{p} = -i\hbar\frac{d}{dx}$ is the momentum operator.

Thus, we conclude that

$$\boxed{\hat{T}(a) = e^{-ia\hat{p}/\hbar}}$$