We derive a formula for the gamma function.

Boas p.538

We will use the fact that $\Gamma(n+1) = n!$

Consider the integral

$$\int_0^\infty e^{-\alpha x}\,dx = \int_0^\infty e^{-\alpha x}\,d(\alpha x)\cdot\frac{1}{\alpha} = \frac{1}{\alpha}.$$

Differentiating both sides with respect to α, we have

$$\int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2} = \frac{1!}{\alpha^2}$$

Again, we have

$$\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{2}{\alpha^3} = \frac{2!}{\alpha^3} \longrightarrow$$

$$\int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{6}{\alpha^4} = \frac{3!}{\alpha^4} \longrightarrow$$

$$\int_0^\infty x^4 e^{-\alpha x}\,dx = \frac{24}{\alpha^5} = \frac{4!}{\alpha^4} \longrightarrow \cdots$$

$$\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}.$$

So, setting $\alpha = 1$, we have

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Using $\Gamma(n+1) = n!$, we derive

$$\boxed{\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx}$$

Of course, we can plug in non-integer values of p, and this function will still be defined in many cases.

<u>Exercise</u> / $\Gamma(p+1) = p\Gamma(p)$.

It is trivial to show this result by integration by parts:

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = e^{-x}x^p\Big|_\infty^0 + \int_0^\infty p x^{p-1} e^{-x}\,dx$$

$$= p \int_0^\infty x^{p-1} e^{-x}\,dx = p\,\Gamma(p).$$

$\blacksquare$