We derive the spherical unit vectors.

In spherical coordinates, we have

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

It follows that the line element ds^2 is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = (dr\sin\theta\cos\phi + r\cos\theta\, d\theta\cos\phi -$$

$$r\sin\theta\sin\phi\, d\phi)^2 + (dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi)^2$$

$$= (dr\cos\theta - r\sin\theta\, d\theta)^2 = \underline{dr^2\sin^2\theta\cos^2\phi} + \underline{dr\sin\theta\cos\phi\, r\cos\theta\, d\theta\cos\phi}$$

$$- \underline{dr\sin\theta\cos\phi\, r\sin\theta\sin\phi\, d\phi} + \underline{r\cos\theta\, d\theta\cos\phi\, dr\sin\theta\cos\phi} + \underline{r^2\cos^2\theta\, d\theta^2\cos^2\phi}$$

$$- \underline{r\cos\theta\, d\theta\cos\phi\, r\sin\theta\sin\phi\, d\phi} - \underline{r\sin\theta\sin\phi\, d\phi\, dr\sin\theta\cos\phi} -$$

$$\underline{r\cos\theta\, d\theta\cos\phi\, r\sin\theta\sin\phi\, d\phi} + \underline{r^2\sin^2\theta\sin^2\phi\, d\phi^2} +$$

$$\underline{dr^2\sin^2\theta\sin^2\phi} + \underline{dr\sin\theta\sin\phi\, r\cos\theta\, d\theta\sin\phi} + \underline{dr\sin\theta\sin\phi\, r\sin\theta\cos\phi\, d\phi}$$

$$+ \underline{r\cos\theta\, d\theta\sin\phi\, dr\sin\theta\sin\phi} + \underline{r^2\cos^2\theta\, d\theta^2\sin^2\phi} + \underline{r\cos\theta\, d\theta\sin\phi\, r\sin\theta\cos\phi\, d\phi}$$

$$+ \underline{r\sin\theta\cos\phi\, d\phi\, dr\sin\theta\sin\phi} + \underline{r\sin\theta\cos\phi\, d\phi\, r\cos\theta\, d\theta\sin\phi} + \underline{r^2\sin^2\theta\cos^2\phi\, d\phi^2}$$

$$+\, dr^2\cos^2\theta - 2r\,dr\cos\theta\sin\theta\, d\theta + r^2\sin^2\theta\, d\theta^2 =$$

$$dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 .$$

Since all the cross-terms vanish, we can say $ds^2 = \vec{ds}\cdot\vec{ds}$, where (assuming the spherical unitvectors are orthonormal) we have

$$\vec{ds} = \hat{a}_r dr + \hat{a}_\theta d\theta + \hat{a}_\phi d\phi$$

and $\hat{r} = \hat{a}_r/|\hat{a}_r|$, $\hat{\theta} = \hat{a}_\theta/|\hat{a}_\theta|$, $\hat{\phi} = \hat{a}_\phi/|\hat{a}_\phi|$.

It follows that $\hat{a}_r\cdot\hat{a}_r = 1 \Rightarrow \hat{a}_r = \hat{r}$.

$$\hat{a}_\theta\cdot\hat{a}_\theta = r^2 \Rightarrow \hat{a}_\theta = r\hat{\theta}$$

$$\hat{a}_\phi\cdot\hat{a}_\phi = r^2\sin^2\theta \Rightarrow \hat{a}_\phi = r\sin\theta\hat{\phi}$$

We also remark that $ds^2 = |dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}|^2 \Longrightarrow$

$$\vec{ds} = (dr\sin\theta\cos\phi + r\cos\theta\, d\theta\cos\phi - r\sin\theta\sin\phi\, d\phi)\hat{x} +$$
$$(dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi)\hat{y} +$$
$$(dr\cos\theta - r\sin\theta\, d\theta)\hat{z}$$

$$= \left(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right) dr +$$

$$\left(r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}\right) d\theta +$$

$$\left(-r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}\right) d\phi \implies$$

$$\boxed{\begin{aligned} \hat{r} &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{aligned}}$$