We prove that the Slater determinant is normalized.

Proof

The Slater determinant is defined as

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(\vec{x}_1) & \cdots & \phi_N(\vec{x}_1) \\ \vdots & \ddots & \vdots \\ \phi_1(\vec{x}_N) & \cdots & \phi_N(\vec{x}_N) \end{vmatrix}$$

$$= \frac{1}{\sqrt{N!}} \sum_{i_1, \dots, i_N = 1}^{N} \varepsilon_{i_1 \cdots i_N} \phi_{i_1}(\vec{x}_1) \cdots \phi_{i_N}(\vec{x}_N),$$

where the final equality comes from the definition of the determinant, and $\langle \phi_i | \phi_j \rangle = \delta_{ij}$.

Then, $\langle \Psi | \Psi \rangle =$

$$\int \prod_{k=1}^{N} d^3 x_k \left[\frac{1}{\sqrt{N!}} \sum_{j_1, \dots, j_N = 1}^{N} \varepsilon_{j_1 \cdots j_N} \phi^*_{j_1}(\vec{x}_1) \cdots \phi^*_{j_N}(\vec{x}_N) \right] \cdot$$

$$\left[\frac{1}{\sqrt{N!}} \sum_{i_1, \dots, i_N = 1}^{N} \varepsilon_{i_1 \cdots i_N} \phi_{i_1}(\vec{x}_1) \cdots \phi_{i_N}(\vec{x}_N) \right] =$$

$$\frac{1}{N!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\,\varepsilon_{i_1\cdots i_N}\int\prod_{k=1}^{N} d^3x_k\;\phi^*_{j_k}(\vec{x}_k)\,\phi_{i_k}(\vec{x}_k) =$$

This is because
$\iint f(x)g(y)\,dx\,dy = \left(\int f(x)dx\right)\left(\int g(y)dy\right)$

$$\frac{1}{N!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\,\varepsilon_{i_1\cdots i_N}\prod_{k=1}^{N}\left(\int d^3x_k\,\phi^*_{j_k}(\vec{x}_k)\phi_{i_k}(\vec{x}_k)\right) =$$

$$\frac{1}{N!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\,\varepsilon_{i_1\cdots i_N}\prod_{k=1}^{N}\delta_{j_k i_k} =$$

$$\frac{1}{N!}\sum_{i_1\cdots i_N}\left(\varepsilon_{i_1\cdots i_N}\right)^2 = \frac{1}{N!}\cdot N! = 1,$$

where the penultimate equality comes from the fact that $|\varepsilon_{i_1\cdots i_N}|=1$ only if $(i_1,\dots,i_N)$ is a permutation of $(1,\dots,N)$, and there are $N!$ such permutations.

$\boxed{Q.E.D.}$