We compute the expectation value of an arbitrary single-particle potential for a system in a Slater determinant state.

The single-particle potential is given by $V_{tot}(\vec{x}_1, \dots, \vec{x}_N) = \sum_{\alpha=1}^{N} V(\vec{x}_\alpha)$

and the Slater determinant is given by

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) = \frac{1}{\sqrt{N!}} \sum_{i_1, \dots, i_N = 1}^{N} \varepsilon_{i_1 \dots i_N} \prod_{k=1}^{N} \phi_{i_k}(\vec{x}_k)$$

It follows that $\left\langle \sum_\alpha V(\vec{x}_\alpha) \right\rangle =$

$$\int \prod_{k=1}^{N} d^3 x_k \left(\frac{1}{\sqrt{N!}} \sum_{j_1 \dots j_N = 1}^{N} \varepsilon_{j_1 \dots j_N} \phi^*_{j_k}(\vec{x}_k) \right) \cdot$$

$$\left(\sum_\alpha V(\vec{x}_\alpha) \right) \left(\frac{1}{\sqrt{N!}} \sum_{i_1, \dots, i_N = 1}^{N} \varepsilon_{i_1 \dots i_N} \prod_{k=1}^{N} \phi_{i_k}(\vec{x}_k) \right) =$$

$$\frac{1}{N!} \sum_{\substack{i_1, \dots, i_N \\ j_1, \dots, j_N}} \varepsilon_{j_1 \dots j_N} \varepsilon_{i_1 \dots i_N} \sum_\alpha \int \prod_{k=1}^{N} d^3 x_k \, \phi^*_{j_k}(\vec{x}_k) V(\vec{x}_\alpha) \phi_{i_k}(\vec{x}_k).$$

We observe that since our total potential $V_{tot}(\vec{x}_1, \dots, \vec{x}_N)$ treats each $\vec{x}_i$ identically, and since the inner product is invariant under the interchange

$\vec{x}_i \longleftrightarrow \vec{x}_j$ for $i \neq j$, $\langle V(\vec{x}_i)\rangle$ cannot differ from $\langle V(\vec{x}_j)\rangle$ in any way. It follows that we only need to compute $\langle V(\vec{x}_i)\rangle$ once, and the end result will be N times that inner product. Without loss of generality, we choose $\langle V(\vec{x}_1)\rangle$. It follows that $\langle V_{tot}\rangle =$

$$\frac{1}{N!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\varepsilon_{i_1\cdots i_N}\cdot N\int\prod_{k=1}^{N} d^3x_k\, \phi^*_{j_k}(\vec{x}_k)\, V(\vec{x}_1)\, \phi_{i_k}(\vec{x}_k)$$

$$=\frac{1}{(N-1)!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\varepsilon_{i_1\cdots i_N}\left(\int d^3x_1\, \phi^*_{j_1}(\vec{x}_1)\, V(\vec{x}_1)\, \phi_{i_1}(\vec{x}_1)\right)\cdot \left(\prod_{k=2}^{N}\int d^3x_k\, \phi^*_{j_k}(\vec{x}_k)\, \phi_{i_k}(\vec{x}_k)\right) =$$

$$\frac{1}{(N-1)!}\sum_{\substack{i_1,\dots,i_N\\ j_1,\dots,j_N}} \varepsilon_{j_1\cdots j_N}\varepsilon_{i_1\cdots i_N}\left(\prod_{k=2}^{N}\delta_{i_k j_k}\right)\left(\int d^3x_1\, \phi^*_{j_1}(\vec{x}_1)\, V(\vec{x}_1)\, \phi_{i_1}(\vec{x}_1)\right)$$

If $\varepsilon_{i_1\cdots i_N}\neq 0$, then the only value of j_1 that will make $\varepsilon_{j_1 i_2\cdots i_N}\neq 0$ is i_1.

$$=\frac{1}{(N-1)!}\sum_{\substack{i_1,\dots,i_N\\ j_1}} \varepsilon_{j_1 i_2\cdots i_N}\varepsilon_{i_1\cdots i_N}\int d^3x_1\, \phi^*_{j_1}(\vec{x}_1)\, V(\vec{x}_1)\, \phi_{i_1}(\vec{x}) =$$

$$\frac{1}{(N-1)!}\sum_{i_1\cdots i_N}\left(\varepsilon_{i_1\cdots i_N}\right)^2\int d^3x\, \phi^*_{i_1}(\vec{x})\, V(\vec{x})\, \phi_{i_1}(\vec{x}) =$$

$$\frac{1}{(N-1)!}\sum_{i_1\cdots i_N}\left(\varepsilon_{i_1\cdots i_N}\right)^2\langle \phi_{i_1}|V|\phi_{i_1}\rangle.$$

Now, fix i_1 and sum over $i_2, \ldots, i_N$. There are $(N-1)!$ nonzero terms in the sum, and none of them affect the value of the inner product. Given that the squared Levi-Civita symbol in each case yields 1, this sum reduces to

$$\frac{1}{(N-1)!} \sum_{i_1=1}^{N} (N-1)! \langle \phi_{i_1} | V | \phi_{i_1} \rangle = \boxed{\sum_i \langle \phi_i | V | \phi_i \rangle}$$

Thus, for a system in a Slater determinant state, the expectation value of an arbitrary single-particle potential is the sum of the expectation values of the single-particle potential with the single-particle wavefunctions.