Exercise 1 We derive the Green's function for the 1D simple harmonic oscillator.

(Source: Boas, p. 461).

The SHO equation of motion is given by

$$\ddot{x} + \omega^2 x = f(t),$$

where $f(t)$ is some forcing. We also impose the BCs

$$x(0) = \dot{x}(0) = 0.$$

We now find the Green's function for this system. We know that if, $Lx = f$ for some differential operator f, then the Green's function will obey

$$L^\dagger G = \delta(t'-t),$$

where $L^\dagger$ is the adjoint of L. Here,

$$L = \frac{d^2}{dt^2} + \omega^2 = L^\dagger,$$

so we can determine the Green's function by solving

$$G_{tt}(t,t') + \omega^2 G(t,t') = \delta(t'-t).$$

subject to the BCs $G(0,t') = G_t(0,t') = 0$.

We derive further boundary conditions by integrating our differential equation over an interval containing $t=t'$:

$$\lim_{\varepsilon\to 0}\int_{t'-\varepsilon}^{t'+\varepsilon} G_{tt}(t,t')\,dt + \underbrace{\lim_{\varepsilon\to 0}\int_{t'-\varepsilon}^{t'+\varepsilon}\omega^2 G(t,t')\,dt}_{\substack{=0 \text{ as we require} \\ G \text{ to be continuous}}} = \underbrace{\lim_{\varepsilon\to 0}\int_{t'-\varepsilon}^{t'+\varepsilon}\delta(t'-t)\,dt}_{=1}$$

so this equation reduces to

$$\Delta G_t(t',t') = 1,$$

which is a third boundary condition, and G being continuous everywhere is a fourth.

We are now ready to solve for the Green's function.

<u>Case 1:</u> $t<t'$

$$G_{tt}(t,t') + \omega^2 G(t,t') = 0 \implies$$
$$G(t,t') = A(t')e^{i\omega t} + B(t')e^{-i\omega t}.$$

Our boundary conditions require

$$\left.\begin{aligned} G(0,t') &= 0 = A(t') + B(t') \\ G_t(0,t') &= 0 = i\omega(A(t') - B(t')) \end{aligned}\right\} \longrightarrow$$

$$G(t,t') = 0, \quad t < t'.$$

Case 2: $t > t'$

$$G_{tt}(t,t') + \omega^2 G(t,t') = 0 \implies$$

$$G(t,t') = C(t')e^{i\omega t} + D(t')e^{-i\omega t}$$

The continuity of G requires that

$$0 = G(t',t') = C(t')e^{i\omega t'} + D(t')e^{-i\omega t'}$$

$$\implies C(t') = -D(t')e^{-2i\omega t'}.$$

The discontinuity of the derivative requires that

$$1 = \Delta G_t(t',t') = G_t(t',t') =$$
$$i\omega(C(t')e^{i\omega t'} - D(t')e^{-i\omega t'}) =$$

$$i\omega\left(-D(t')e^{-i\omega t'} - D(t')e^{-i\omega t'}\right) \implies$$

$$D(t') = \frac{i}{2\omega} e^{i\omega t'} \implies$$

$$G(t,t') = -\frac{i}{2\omega} e^{-i\omega t'} e^{i\omega t} + \frac{i}{2\omega} e^{i\omega t'} e^{-i\omega t}$$

$$= \frac{i}{2\omega}\left(-e^{i\omega(t-t')} + e^{-i\omega(t-t')}\right) =$$

$$\frac{i}{2\omega} \cdot -2i \sin\omega(t-t') = \frac{\sin\omega(t-t')}{\omega}, \quad t > t'.$$

Thus, we conclude that the Green's function for the 1D SHO is given by

$$\boxed{G(t,t') = \begin{cases} 0, & t < t' \\ \frac{\sin\omega(t-t')}{\omega}, & t > t' \end{cases}}$$

And the solution to the equation with forcing $f(t)$ is

$$x(t) = \int_0^{\infty} G(t,t') f(t')\, dt'.$$