Exercise/ We derive the simple harmonic oscillator algebra:

This note is a bit disorganized. Sorry.

$$[a, a^\dagger] = 1, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle,$$

$$a^\dagger|n-1\rangle = \sqrt{n}\,|n\rangle.$$

To prove the first commutator, we use the definition from Griffiths Eq. 2.48:

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip)$$

We apply these to a test function

$$[a, a^\dagger]f = \frac{1}{2m\hbar\omega}\left(a a^\dagger f - a^\dagger a f\right) =$$

$$\frac{1}{2m\hbar\omega}\Big[\big((m\omega x)^2 - im\omega(xp - px) + p^2\big)f -$$

$$\big((m\omega x)^2 + im\omega(xp - px] + p^2\big)f\Big]$$

$$= -\frac{im\omega}{m\hbar\omega}[x, p]f = -\frac{i}{\hbar}\left[x\cdot(-i\hbar f') - (-i\hbar f - i\hbar x f')\right]$$

$= -\frac{i^2 \hbar}{\hbar} f = f$. Thus, we have shown $[a, a^\dagger] = 1$. $\boxed{/\!/}$

Next, we wish to show $a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$, $a|n\rangle = \sqrt{n}\, |n-1\rangle$.

First, we must show that a and $a^\dagger$ are Hermitian conjugates of one another:

$$\langle f | a^\dagger g \rangle = \langle af | g \rangle, \text{ and } \langle f | ag \rangle = \langle a^\dagger f | g \rangle.$$

Expaning the definitions of $a^\dagger$ and a fully, we have

$$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip) = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)$$

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip) = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x + \hbar\frac{d}{dx}\right)$$

Placing both in an inner product, we have

$$\int_{-\infty}^{\infty} f^* \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x \mp \hbar\frac{d}{dx}\right) g\, dx =$$

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$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2m\hbar\omega}} \left(f^* m\omega x g \mp f^* \hbar \frac{dg}{dx} \right) dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega x f^* g \pm \hbar \frac{df^*}{dx} g \right) dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega x \pm \hbar \frac{d}{dx} \right) f^* g \, dx .$$

We have thus simultaneously shown that $\langle f | a^\dagger g \rangle = \langle a f | g \rangle$ and $\langle f | a g \rangle = \langle a^\dagger f | g \rangle$, so a and $a^\dagger$ are indeed Hermitian conjugates. $\boxtimes$

Next, we must show that the raising operators map from one eigenstate of the SHO to another.

Consider that

$$a a^\dagger = \frac{1}{2m\hbar\omega} \left((m\omega x)^2 - i m\omega (xp - px) + p^2 \right)$$

$$= \frac{1}{2m\hbar\omega} \left((m\omega x)^2 + m\omega\hbar + p^2 \right) \Rightarrow \hat{H} = \hbar\omega \left(a a^\dagger - \tfrac{1}{2} \right),$$

since the Hamiltonian is simply $\hat{H} = \frac{1}{2} m\omega^2 x^2 + p^2/2m$

Consider an eigenstate $|E\rangle$ with energy E. Then

$$\hat{H}|E\rangle = E|E\rangle \Rightarrow a^\dagger\hat{H}|E\rangle = a^\dagger E|E\rangle = E a^\dagger|E\rangle =$$

$$[a^\dagger, H]|E\rangle + \hat{H}a^\dagger|E\rangle = \hbar\omega\big(a[a^\dagger, a^\dagger] + [a^\dagger, a]a^\dagger -$$

$$[a^\dagger, 1/2]\big)|E\rangle + \hat{H}a^\dagger|E\rangle = -\hbar\omega a^\dagger|E\rangle + \hat{H}a^\dagger|E\rangle \Leftrightarrow$$

$\hat{H}a^\dagger|E\rangle = (E + \hbar\omega)a^\dagger|E\rangle$, which shows that $a^\dagger|E\rangle$ is also an eigenvector of $\hat{H}$. It follows that successively applying the raising operator to the ground state will get us to a ladder of eigenstates

$$|0\rangle \rightarrow A_1|1\rangle = a^\dagger|0\rangle \rightarrow A_2|2\rangle = (a^\dagger)^2|0\rangle \rightarrow \cdots$$

We now introduce the number operator. We remark that

$$a^\dagger a = \frac{1}{2m\hbar\omega}(m\omega x - ip)(m\omega x + ip) =$$

$$\frac{1}{2m\hbar\omega}\left((m\omega x)^2 + im\omega(xp - px) + p^2\right) \Rightarrow \hat{H} = \hbar\omega(a^\dagger a + 1/2)$$

$\equiv \hbar\omega(\hat{N} + 1/2)$, where $\hat{N}$ is the number operator. We will need the spectrum of the number operator. We just showed that

$$\hat{H}(a^+)^n|0\rangle = (E_0 + n\hbar\omega)(a^+)^n|0\rangle,$$

so let us now determine the ground state energy of the SHO.

We remark that, just as a^+ raises the energy of a state by $\hbar\omega$, a lowers the energy of a state by $\hbar\omega$.

$$\hat{H}|E\rangle = E|E\rangle = a\hat{H}|E\rangle = Ea|E\rangle = [a,\hat{H}]|E\rangle + \hat{H}a|E\rangle$$

$$= \hbar\omega[a,a^+]a|E\rangle + \hat{H}a|E\rangle = \hbar\omega a|E\rangle + \hat{H}a|E\rangle \iff$$

$$\hat{H}a|E\rangle = (E-\hbar\omega)a|E\rangle.$$

We argue that there must exist a state $|0\rangle$ such that $a|0\rangle = 0$. If we continue to lower the energy, at some point, we will reach a state with $E < V_{min} \implies$

$$-\frac{\hbar^2}{2m}\psi'' = (E - V(x))\psi \iff \frac{\hbar^2}{2m}\psi'' = K(x)^2\psi, \text{ where}$$

$K(x)^2 > 0 \quad \forall x$. This function will either be concave-up or concave-down at all points, and thus cannot be normalizable, and thus must not be a physical state.

We now solve $\langle x | a | 0 \rangle = 0 = (m\omega x + \hbar \, d/dx)\psi \implies$

$$\psi' = -\frac{m\omega}{\hbar} x \psi \implies \ln\psi = C - m\omega x^2/2\hbar \implies$$

$\psi = A e^{-m\omega x^2/2\hbar}$, and normalization requires $A = \left(\frac{\pi\hbar}{m\omega}\right)^{1/4}$

$\implies \psi = \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} e^{-m\omega x^2/2\hbar}$. We can find the energy of this state using the Schrödinger equation

$$\frac{-\hbar^2}{2m}\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\frac{d}{dx}\left(e^{-m\omega x^2/2\hbar}\cdot\frac{-m\omega x}{\hbar}\right) + \frac{1}{2}m\omega^2 x^2\psi$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\left(e^{-m\omega x^2/2\hbar}\cdot\left(\frac{-m\omega x}{\hbar}\right)^2 - \frac{m\omega}{\hbar}e^{-m\omega x^2/2\hbar}\right)$$

$$+ \frac{1}{2}m\omega^2 x^2\psi = -\frac{\hbar^2}{2m}\psi\left(-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right) + \frac{1}{2}m\omega^2 x^2\psi$$

$$= \frac{\hbar\omega}{2}\psi = E_0\psi \iff \boxed{E_0 = \frac{\hbar\omega}{2}}$$

It follows that the SHO spectrum is given by

$$\boxed{E_n = \hbar\omega(n + 1/2)}$$

We now easily determine the spectrum of the number operator:

$$\hat{H}|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle = \hbar\omega\left(\hat{N}+\tfrac{1}{2}\right)|n\rangle \implies$$

$\hat{N}|n\rangle = n|n\rangle$. We can also see that the eigenvectors of $\hat{N}$ are the same as those of $\hat{H}$.

We remark that $a\hat{N}|n\rangle = n\,a|n\rangle = a\,a^\dagger a|n\rangle$

$$= [a, a^\dagger]a|n\rangle + a^\dagger a\, a|n\rangle = a|n\rangle + \hat{N}a|n\rangle \iff$$

$$\hat{N}a|n\rangle = (n-1)a|n\rangle,$$

and

$$a^\dagger\hat{N}|n\rangle = n\,a^\dagger|n\rangle = a^\dagger a^\dagger a|n\rangle = a^\dagger[a^\dagger, a]|n\rangle +$$

$$a^\dagger a\,a^\dagger|n\rangle = -a^\dagger|n\rangle + \hat{N}a^\dagger|n\rangle \iff$$

$$\hat{N}a^\dagger|n\rangle = (n+1)a^\dagger|n\rangle.$$

We also argued above that $a^\dagger|n\rangle \propto |n+1\rangle$, $a|n\rangle \propto |n-1\rangle$.

With these pieces, we can finally finish deriving the SHO algebra.

Suppose $a^\dagger|n\rangle = c|n+1\rangle$. Then (using the fact that $a = (a^\dagger)^\dagger$),

$$\langle n|aa^\dagger|n\rangle = \langle n+1|c^* c|n+1\rangle \implies$$

$$\langle n|[a,a^\dagger] + a^\dagger a|n\rangle = |c|^2 = \langle n|n\rangle + \langle n|\hat{N}|n\rangle$$

$$= n+1 \implies c = \sqrt{n+1}$$ (where the lack of complex phase is a convention).

Similarly,

$$a|n\rangle \equiv d|n-1\rangle \implies \langle n|a^\dagger a|n\rangle = \langle n-1|d^* d|n-1\rangle$$

$$\implies n = |d|^2 \implies d = \sqrt{n},$$ so we arrive at

two more equations for our algebra

$$\boxed{\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a|n\rangle &= \sqrt{n}\,|n\rangle \end{aligned}}$$

Finally, if we start from our ground state, we get

$$|0\rangle \mapsto a^\dagger|0\rangle = |1\rangle \mapsto a^\dagger|1\rangle = \sqrt{2}\,|2\rangle \mapsto a^\dagger|2\rangle =$$

$$\sqrt{3}\,|3\rangle \Rightarrow |3\rangle = \frac{1}{\sqrt{3}} a^{+}|2\rangle = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} (a^{+})^{2}|1\rangle$$

$$= \frac{1}{\sqrt{3!}} (a^{+})^{3}|0\rangle \longmapsto a^{+}|3\rangle = \sqrt{4}\,|4\rangle \Rightarrow$$

$$|4\rangle = \frac{1}{\sqrt{4}} a^{+}|3\rangle = \frac{1}{\sqrt{4!}} (a^{+})^{4}|0\rangle \longmapsto \cdots$$

Suppose $|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle$. Then,

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \Rightarrow |n+1\rangle = \frac{1}{\sqrt{n+1}} a^{+}|n\rangle$$

$$= \frac{(a^{+})^{n+1}}{\sqrt{(n+1)!}}|0\rangle,$$

and so we have shown by induction that

$$\boxed{|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle}$$

$\square$

Just go read Griffiths!