We derive the relativistic energy for a particle.

The proper time $d\tau$ is defined as

$$d\tau = \sqrt{-ds^2} = \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} =$$

$$\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}\, dt = \sqrt{\left(\frac{dx^0}{dt}\right)^2 - \left(\frac{d\vec{x}}{dt}\right)^2}\, dt =$$

$$\sqrt{\left(\frac{dt}{dt}\right)^2 - \vec{v}(t)^2}\, dt = \sqrt{1-\vec{v}(t)^2}\, dt$$

The four velocity is defined as $u^\mu = \dfrac{dx^\mu}{d\tau} = \dfrac{1}{\sqrt{1-\vec{v}(t)^2}} \dfrac{dx^\mu}{dt} =$

$$\gamma\left(\frac{dt}{dt}, \frac{d\vec{x}}{dt}\right) = \gamma(1, \vec{v}).$$

We remark that the 4-velocity's inner product with itself is -1:

$$\eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left(\eta_{\mu\nu}\, dx^\mu dx^\nu\right) \cdot \frac{1}{d\tau^2} =$$

$$\frac{-(d\tau)^2}{(d\tau)^2} = -1.$$

We now construct the 4-momentum

$p^\mu = m U^\mu = (\gamma m, \gamma m \vec{v})$, where m is the particle's rest mass.

However, we recognize the relativistic energy and momentum here, so

$$p^\mu = (E, \vec{p}).$$

Finally, we compute the inner product of the 4-momentum with itself:

$$\eta_{\mu\nu} p^\mu p^\nu = -(p^0)^2 + \vec{p}^{\,2} = -E^2 + \vec{p}^{\,2}$$
$$\parallel$$
$$\eta_{\mu\nu} m U^\mu m U^\nu = m^2 \eta_{\mu\nu} U^\mu U^\nu = -m^2 .$$

Together, this implies the relativistic energy formula

$$\boxed{E^2 = m^2 + \vec{p}^{\,2}}$$