Exercise We derive the radial Schrödinger equation for a spherically symmetric system.

The Schrödinger equation for a spherically symmetric system is given by

$$-\frac{\hbar^2}{2m}\Delta\Psi + V(r)\Psi = E\Psi$$

We separate variables: $\Psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$.

We also use the definition of the Laplacian in spherical coordinates:

$$\Delta(RY) = \frac{1}{r^2}\partial_r\left(r^2\partial_r(RY)\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta(RY)\right)$$

$$+\frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}RY = \frac{2}{r}R'Y + R''Y + \frac{\cot\theta}{r^2}RY_\theta$$

$$+\frac{1}{r^2}RY_{\theta\theta} + \frac{1}{r^2\sin^2\theta}RY_{\phi\phi}.$$

Plugging into the Schrödinger equation, we have

$$-\frac{\hbar^2}{2m}\left[\frac{2}{r}R'Y + R''Y + \frac{\cot\theta}{r^2}RY_\theta + \frac{1}{r^2}RY_{\theta\theta} + \frac{1}{r^2\sin^2\theta}RY_{\phi\phi}\right]$$

$$+V(r)RY = ERY \iff$$

$$-\frac{\hbar^2}{2m}\left[\frac{2}{r}\frac{R'}{R}+\frac{R''}{R}+\frac{\cot\theta}{r^2}\frac{Y_\theta}{Y}+\frac{1}{r^2}\frac{Y_{\theta\theta}}{Y}+\frac{1}{r^2\sin^2\theta}\frac{Y_{\phi\phi}}{Y}\right]$$

$$+V(r)-E=0 \iff$$

$$2r\,R'/R+r^2R''/R+\cot\theta\,Y_\theta/Y+Y_{\theta\theta}/Y+\frac{1}{\sin^2\theta}Y_{\phi\phi}/Y$$

$$-\frac{2mr^2}{\hbar^2}(V(r)-E)=0.$$

We can now group these results as follows:

$$\left[2r\frac{R'(r)}{R(r)}+r^2\frac{R''(r)}{R(r)}-\frac{2mr^2}{\hbar^2}(V(r)-E)\right]+$$

$$\left[\cot\theta\frac{Y_\theta(\theta,\phi)}{Y(\theta,\phi)}+\frac{Y_{\theta\theta}(\theta,\phi)}{Y(\theta,\phi)}+\frac{1}{\sin^2\theta}\frac{Y_{\phi\phi}(\theta,\phi)}{Y(\theta,\phi)}\right]=0$$

Since the first group only depends on r, and the second on θ and ϕ, we conclude that they are both constant, with opposite signs. We call the separation constant $l(l+1)$. Just looking at the radial equation, we have

$$2r\frac{R'(r)}{R(r)}+r^2\frac{R''(r)}{R(r)}-\frac{2mr^2}{\hbar^2}(V(r)-E)=l(l+1)\iff$$

$$2rR' + r^2R'' - \frac{2mr^2R}{\hbar^2}(V(r) - E) = l(l+1)R.$$

Let $u = rR \Rightarrow u' = R + rR' \Rightarrow u'' = 2R' + rR''$.

Then, this equation becomes

$$ru'' - \frac{2mr}{\hbar^2}u(V(r) - E) = l(l+1)u/r \iff$$

$$-\frac{\hbar^2}{2m}u'' + u(V(r) - E) = \frac{-\hbar^2}{2mr^2}l(l+1)u \iff$$

$$\boxed{-\frac{\hbar^2}{2m}u'' + \left(V(r) + \frac{\hbar^2}{2mr^2}l(l+1)\right)u = Eu}$$

This is the radial Schrödinger equation.