We derive the product rule from the definition of the derivative.

Let $h(x) = f(x)g(x)$. Then,

$$h'(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(h(x+\varepsilon) - h(x)\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(f(x+\varepsilon)g(x+\varepsilon) - f(x)g(x)\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[f(x+\varepsilon)g(x+\varepsilon) - f(x)g(x+\varepsilon) + f(x)g(x+\varepsilon) - f(x)g(x)\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[g(x+\varepsilon)\left(f(x+\varepsilon) - f(x)\right) + f(x)\left(g(x+\varepsilon) - g(x)\right)\right]$$

$$= \lim_{\varepsilon \to 0} g(x+\varepsilon)\frac{1}{\varepsilon}\left(f(x+\varepsilon) - f(x)\right) + f(x)\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(g(x+\varepsilon) - g(x)\right) =$$

$$\boxed{g(x)f'(x) + f(x)g'(x)}$$