The Path Integral in Quantum Mechanics

Matt Kafker

Exercise:

We derive the path integral in quantum mechanics.

Solution:

Consider a particle in 1D with Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x)$$

where $[x, p] = i$. We wish to calculate the amplitude A that the system starts at position x_i at time t_0 and ends at position x_f at time t_f.

$$A = \langle x_f | e^{-iH(t_f - t_0)} | x_i \rangle$$

But we cannot calculate this for arbitrary (potentially time-dependent) Hamiltonians. Instead, we discretize time into small chunks δt. Then the amplitude describing the particle moving in the sequence of positions $x_i \to x_1 \to \cdots \to x_n \to x_f$ is given by

$$\langle x_f | e^{-iH\delta t} | x_n \rangle \langle x_n | \cdots | x_1 \rangle \langle x_1 | e^{-iH\delta t} | x_0 \rangle$$

These intermediate positions $x_1 \ldots x_n$ are arbitrary, so to get A we integrate over them all (equivalently, we have simply inserted completeness relations)

$$A = \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle x_f | e^{-iH\delta t} | x_n \rangle \langle x_n | \cdots | x_1 \rangle \langle x_1 | e^{-iH\delta t} | x_0 \rangle.$$

We focus on an individual term

$$A_{21} = \langle x_2 | e^{-iH\delta t} | x_1 \rangle = \langle x_2 | e^{-i\frac{p^2}{2m}\delta t - iV(x)\delta t} | x_1 \rangle.$$

We recall that if $[A,B]\neq 0$, $e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}e^{\text{nested commutators}}\cdots$. Furthermore, we consider $\delta t\ll 1$, so we can simplify this expression to

$$
\begin{gathered}
A_{21}\approx \langle x_2|e^{-i\frac{p^2}{2m}\delta t}e^{-iV(x)\delta t}|x_1\rangle = \\
e^{-iV(x_1)\delta t}\langle x_2|e^{-i\frac{p^2}{2m}\delta t}|x_1\rangle = e^{-iV(x_1)\delta t}\int_{-\infty}^{\infty}dp\langle x_2|e^{-i\frac{p^2}{2m}\delta t}|p\rangle\langle p|x_1\rangle = \\
e^{-iV(x_1)\delta t}\int_{-\infty}^{\infty}dpe^{-i\frac{p^2}{2m}\delta t}\langle x_2|p\rangle\langle p|x_1\rangle = e^{-iV(x_1)\delta t}\int_{-\infty}^{\infty}dpe^{-i\frac{p^2}{2m}\delta t}e^{ip(x_2-x_1)} = \\
Ne^{-iV(x_1)\delta t}e^{i\frac{m(x_2-x_1)^2}{2\delta t^2}\delta t}\approx Ne^{-iV(x_1)\delta t}e^{i\frac{m\dot{x}_1^2}{2}\delta t}=Ne^{iL(x_1,\dot{x}_1)\delta t}.
\end{gathered}
$$

The full amplitude then becomes

$$
A=N^n\int\prod_j dx_je^{iL(x_n,\dot{x}_n)\delta t+\cdots+iL(x_1,\dot{x}_1)\delta t}
$$

and in the limit that $\delta t\to 0$, we have

$$
A=\bar{N}\int_{x(t_0)=x_i}^{x(t_f)=x_f}\underbrace{\mathcal{D}x(t)}_{\text{all possible paths}}e^{i\int dtL(x,\dot{x})}=\boxed{\bar{N}\int_{x(t_0)=x_i}^{x(t_f)=x_f}\mathcal{D}x(t)e^{iS}.}
$$