

Exercise We derive the orthogonality of sine and cosine.

We consider the periodic functions $\sin(n\pi x/L)$, $\cos(m\pi x/L)$ defined on $[0, 2L]$, where $n, m \in \mathbb{N}$.

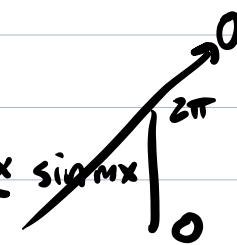
We first evaluate

$2L$

$$\int \sin(n\pi x/L) \sin(m\pi x/L) dx =$$

$$\frac{L}{\pi} \int_0^{2\pi} \sin nx \sin mx dx$$

$$\stackrel{\text{IBP}}{=} \frac{L}{\pi} \left[-\frac{\cos nx}{n} \sin mx \right]_0^{2\pi}$$



$$+ \left[\frac{\cos nx}{n} \cdot m \cos mx dx \right] \stackrel{\text{IBP}}{=} \frac{L}{\pi} \frac{m}{n} \left[\frac{\sin nx}{n} \cos mx \right]_0^{2\pi}$$

$$+ \left[\frac{\sin nx}{n} m \sin mx \right] = \frac{L}{\pi} \frac{m^2}{n^2} \int_0^{2\pi} \sin nx \sin mx dx$$

$$= \frac{m^2}{n^2} \int_0^{2L} \sin(n\pi x/L) \sin(m\pi x/L) dx \implies$$

$$\left(\int_0^{2L} \sin(n\pi x/L) \sin(m\pi x/L) dx \right) \left(1 - m^2/n^2 \right) = 0.$$

We treat two cases.

Case 1: $m = n$.

Then, $1 - m^2/n^2 = 0$, and the equality above is satisfied. However, we don't know anything about the integral. The trig identity

$$\sin^2(z) = \frac{1}{2} (1 - \cos(2z))$$

allows us to compute the integral directly

$$\int_0^{2L} \sin^2(n\pi x/L) dx = L - \int_0^{2L} \cos(2\pi n x/L) dx$$

$$= L - \frac{L}{2\pi n} \int_0^{4\pi n} \cos(2\pi n x/L) d(2\pi n x/L)$$

$$= L - \frac{L}{2\pi n} (\sin(4\pi n) - \sin 0) = L.$$

Case 2: $m \neq n$.

Then, $1 - m^2/n^2 \neq 0$, so it must be the case that

$$\int_0^{2L} \sin(n\pi x/L) \sin(m\pi x/L) dx = 0.$$

Combining these results, we get

$$\boxed{\int_0^{2L} \sin(n\pi x/L) \sin(m\pi x/L) dx = L S_{nm}}$$

Next, we compute

$$\int_0^{2L} \cos(n\pi x/L) \cos(m\pi x/L) dx =$$

$$\frac{L}{\pi} \int_0^{2\pi} \cos(nx) \cos(mx) dx \stackrel{\text{IBP}}{=} \frac{L}{\pi} \left[\frac{\sin nx}{n} \cos mx \right]_0^{2\pi} +$$

$$\left. \frac{L}{\pi} \frac{m}{n} \sin mx \right] \stackrel{\text{IBP}}{=} \frac{L}{\pi} \frac{m}{n} \left[\frac{-\cos nx}{n} \sin mx \right]_0^{2\pi} +$$

$$\left[\int_0^{2\pi} \frac{\cos nx}{n} m \cos mx dx \right] = \frac{L}{\pi} \frac{m^2}{n^2} \int_0^{2\pi} \cos nx \cos mx dx$$

$$= \frac{m^2}{n^2} \int_0^{2L} \cos(n\pi x/L) \cos(m\pi x/L) dx \implies$$

$$\left(\int_0^{2L} \cos(n\pi x/L) \cos(m\pi x/L) dx \right) \left(1 - m^2/n^2 \right) = 0.$$

Once again, we consider two cases.

Case 1: $m=n$. The equality above is satisfied, and we have

$$\int_0^{2L} \cos^2(n\pi x/L) dx = \int_0^{2L} \frac{1}{2} (1 + \cos(2n\pi x/L)) dx$$

$$= L.$$

Case 2: $m \neq n$.

To satisfy the equality above, we require

$$\int_0^{2L} \cos(n\pi x/L) \cos(m\pi x/L) dx = 0.$$

Combining these two results, we have

$$\boxed{\int_0^{2L} \cos(n\pi x/L) \cos(m\pi x/L) dx = L \delta_{nm}}$$

Finally, we consider

$$\int_0^{2L} \sin(n\pi x/L) \cos(m\pi x/L) dx \stackrel{IBP}{=}$$

$$\sin(n\pi x/L) \frac{L}{m\pi} \Big|_0^{2L} - \cancel{\int_0^{2L} \sin(m\pi x/L) \cos(n\pi x/L) dx}$$

$$\frac{n}{m} \int_0^{2L} \cos(n\pi x/L) \sin(m\pi x/L) dx \stackrel{IBP}{=}$$

$$\frac{n}{m} \left[\cos(n\pi x/L) \left(-\frac{L}{m\pi} \right) \cos(m\pi x/L) \right]_0^{2L} -$$

$$\frac{n}{m} \int_0^{2L} \sin\left(\frac{n\pi x}{L}\right) \cos(m\pi x/L) dx =$$

$$\frac{n}{m} \left[-\frac{n}{m} \int_0^{2L} \sin(n\pi x/L) \cos(m\pi x/L) dx \right]$$

$$\Rightarrow \left(\int_0^{2L} \sin(n\pi x/L) \cos(m\pi x/L) dx \right) \left(1 + \frac{n^2}{m^2} \right) = 0 \Rightarrow$$

$\boxed{\int_0^{2L} \sin(n\pi x/L) \cos(m\pi x/L) dx = 0}$

since $1 + n^2/m^2 > 1$.