Exercise | We derive the orthogonality of sine and cosine.

We consider the periodic functions $\sin(n\pi x/L)$, $\cos(n\pi x/L)$ defined on $[0,2L]$, where $n,m \in \mathbb{N}$.

We first evaluate

$$\int_0^{2L} \sin(n\pi x/L)\,\sin(m\pi x/L)\,dx =$$

$$\frac{L}{\pi}\int_0^{2\pi} \sin nx \,\sin mx\,dx \overset{\text{IBP}}{=} \frac{L}{\pi}\Bigg[\cancelto{0}{-\frac{\cos nx}{n}\,\sin mx\,\Big|_0^{2\pi}}$$

$$+\int_0^{2\pi} \frac{\cos nx}{n}\cdot m\cos mx\,dx\Bigg] \overset{\text{IBP}}{=} \frac{L}{\pi}\frac{m}{n}\Bigg[\cancelto{0}{\frac{\sin nx}{n}\cos mx\,\Big|_0^{2\pi}}$$

$$+\int_0^{2\pi} \frac{\sin nx}{n}\, m\sin mx\Bigg] = \frac{L}{\pi}\frac{m^2}{n^2}\int_0^{2\pi} \sin nx\,\sin mx\,dx$$

$$= \frac{m^2}{n^2}\int_0^{2L} \sin(n\pi x/L)\,\sin(m\pi x/L)\,dx \implies$$

$$\left(\int_0^{2L} \sin(n\pi x/L)\sin(m\pi x/L)\,dx\right)(1-m^2/n^2) = 0.$$

We treat two cases.

Case 1: $m=n$.

Then, $1-m^2/n^2 = 0$, and the equality above is satisfied. However, we don't know anything about the integral. The trig identity

$$\sin^2(z) = \frac{1}{2}(1-\cos(2z))$$

allows us to compute the integral directly

$$\int_0^{2L} \sin^2(n\pi x/L)\,dx = L - \int_0^{2L} \cos(2\pi n x/L)\,dx$$

$$= L - \frac{L}{2\pi n}\int_0^{4\pi n} \cos(2\pi n x/L)\,d(2\pi n x/L)$$

$$= L - \frac{L}{2\pi n}(\sin(4\pi n) - \sin 0) = L.$$

Case 2: $m \neq n$.

Then, $1 - m^2/n^2 \neq 0$, so it must be the case that

$$\int_0^{2L} \sin(n\pi x/L)\,\sin(m\pi x/L) = 0.$$

Combining these results, we get

$$\boxed{\int_0^{2L} \sin(n\pi x/L)\sin(m\pi x/L)\,dx = L\,\delta_{nm}}$$

Next, we compute

$$\int_0^{2L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx =$$

$$\frac{L}{\pi}\int_0^{2\pi} \cos(nx)\cos(mx)\,dx \overset{\text{IBP}}{=} \frac{L}{\pi}\left[\cancel{\frac{\sin nx}{n}\cos mx\Big|_0^{2\pi}}^{\,0} + \right.$$

$$\left.\int_0^{2\pi} \frac{\sin nx}{n}\, m \sin mx\,dx\right] \overset{\text{IBP}}{=} \frac{L}{\pi}\frac{m}{n}\left[\frac{-\cos nx}{n}\sin mx\Big|_0^{2\pi} + \right.$$

$$\int_0^{2\pi} \frac{\cos nx}{n} \, m\cos mx \, dx \Big] = \frac{L}{\pi}\frac{m^2}{n^2}\int_0^{2\pi} \cos nx \cos mx \, dx$$

$$= \frac{m^2}{n^2}\int_0^{2L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx \implies$$

$$\left(\int_0^{2L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx\right)\left(1-m^2/n^2\right) = 0.$$

Once again, we consider two cases.

<u>Case 1:</u> $m=n$. The equality above is satisfied, and we have

$$\int_0^{2L} \cos^2(n\pi x/L)\,dx = \int_0^{2L} \frac{1}{2}\left(1+\cos(2n\pi x/L)\right)dx$$

$$= L.$$

<u>Case 2:</u> $m \neq n$.

To satisfy the equality above, we require

$$\int_0^{2L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx = 0.$$

Combining these two results, we have

$$\boxed{\int_0^{2L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx = L\delta_{nm}}$$

Finally, we consider

$$\int_0^{2L} \sin(n\pi x/L)\cos(m\pi x/L)\,dx \overset{\text{IBP}}{=}$$

$$\sin(n\pi x/L)\frac{L}{m\pi}\underbrace{\cancel{\sin(m\pi x/L)\Big|_0^{2L}}}_{0} -$$

$$\frac{n}{m}\int_0^{2L} \cos(n\pi x/L)\sin(m\pi x/L)\,dx \overset{\text{IBP}}{=}$$

$$\frac{n}{m}\left[\cos n\pi x/L\left(-\frac{L}{m\pi}\right)\cos(m\pi x/L)\Big|_0^{2L} -\right.$$

$$\left.\frac{n}{m}\int_0^{2L}\sin\left(\frac{n\pi x}{L}\right)\cos(m\pi x/L)\,dx\right] =$$

$$\frac{n}{m}\left[-\frac{n}{m}\int_0^{2L}\sin(n\pi x/L)\cos(m\pi x/L)\,dx\right]$$

$$\Rightarrow \left(\int_0^{2L}\sin n\pi x/L\,\cos m\pi x/L\,dx\right)\left(1+n^2/m^2\right) = 0 \Rightarrow$$

$$\boxed{\int_0^{2L}\sin n\pi x/L\,\cos m\pi x/L\,dx = 0}$$

since $1+n^2/m^2 > 1$.