6.11

Did not attempt.

6.2

We consider a simple network (i.e. no multi-edges or self-loops) of n nodes and one component. That is, the network is connected.

We know that the minimum number of edges it can have while remaining simple and connected is $\boxed{m_{min} = n-1.}$

The proof is given on page 123. Such a network is a tree.

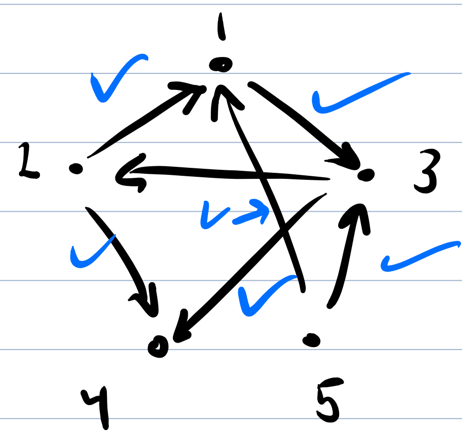
The maximum number of edges will occur when there is an undirected edge between every two nodes. As the edges are undirected, m_{max} is given by the number of ways one can choose two elements from a set of n where order is irrelevant, and we are sampling without replacement:

$$\boxed{m_{max} = \binom{n}{2} = \frac{n!}{2!(n-2)!}}$$

This is a complete graph.

6.3

a.) We construct the adjacency matrix of the network



We recall that the adjacency matrix is given by

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from } j \text{ to } i \\ 0 & \text{otherwise} \end{cases}$$

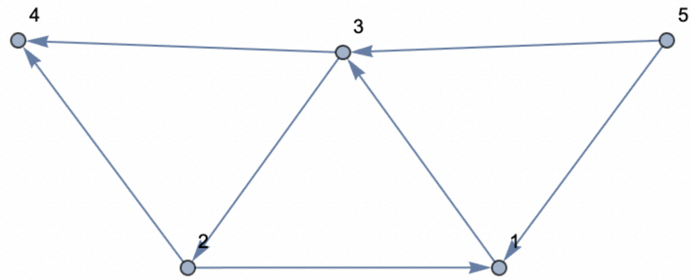
Thus, we have

$$\boxed{A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

We confirm that this is correct in Mathematica:

In[24]:=
```
Graph[{2 ↔ 1, 1 ↔ 3, 5 ↔ 3, 3 ↔ 4, 3 ↔ 2, 2 ↔ 4, 5 ↔ 1}, VertexLabels → "Name"]
```

Out[24]=

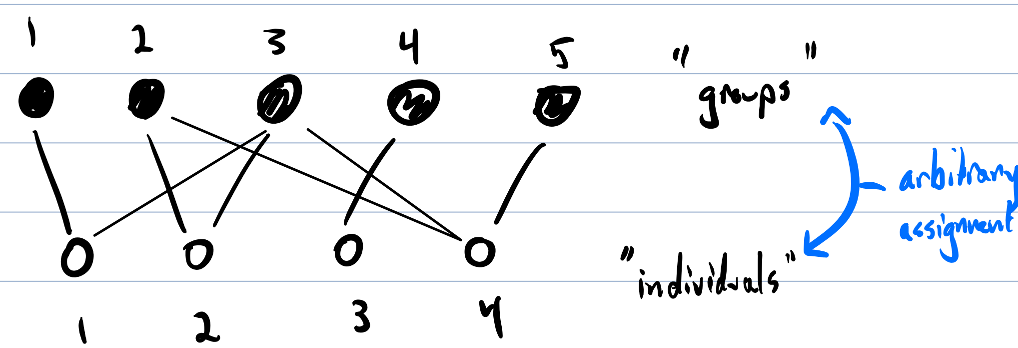


In[26]:=
```
AdjacencyMatrix[%24] // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b) We find the incidence matrix of the bipartite graph



We recall that the incidence matrix B is defined as

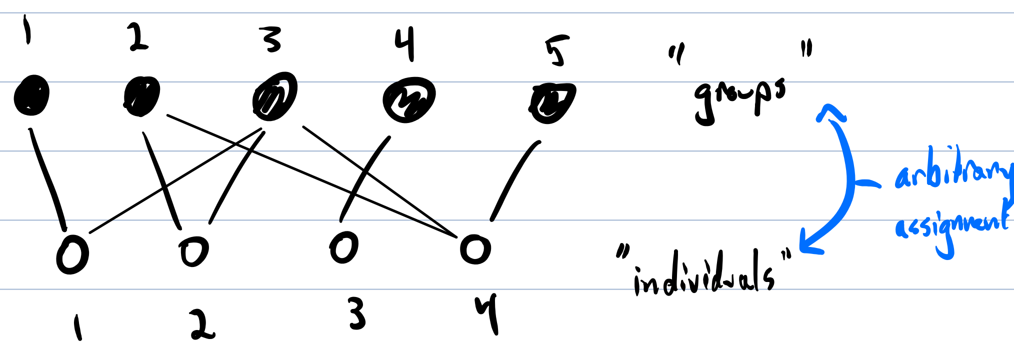
$$B_{ij} = \begin{cases} 1 & \text{if item } j \text{ belongs to group } i \\ 0 & \text{otherwise} \end{cases}$$

Following Figure 6.4b, our matrix will be 5×4:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We remark that if we had labeled the groups and individuals the other way, we would have derived the transpose of this matrix.

c.) We derive the projection matrix of the graph



onto its black nodes (i.e. onto the groups). As there are 5 groups, our projection matrix will be 5×5.

The two different projections are $P = B^T B$ (4×4) and $P' = BB^T$ (5×5). Thus, the projection matrix onto the black nodes is given by

$$P' = BB^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The off-diagonal elements of this matrix, P'_{ij}, give the number of common members of groups i and j. The diagonal elements, P'_{ii}, give the number of elements of group i.

We confirm this result by observing the original bipartite graph.

6.4

Let A be the adjacency matrix, and $\vec{1}$ be the vector of all 1's.

a.) We know that

$$k_i = \sum_j A_{ij} = \vec{A}_i \cdot \vec{1}, \text{ where } \vec{A}_i = \text{Table}[A_{ij}, \{j, n\}]$$

$$\Longrightarrow \boxed{\vec{k} = A\vec{1}}$$

b.) We know that the number of edges, m, obeys

$$2m = \sum_i k_i = \vec{k} \cdot \vec{1} = (A\vec{1}) \cdot \vec{1} =$$

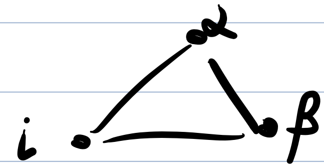
$$\vec{1}^T A \vec{1} \Rightarrow \boxed{m = \frac{1}{2} \vec{1}^T A \vec{1}}$$

c.) We express N, where N_{ij} is # of common neighbors of i and j. A common neighbor is a node α such that α and i are neighbors, and α and j are neighbors. Thus, N_{ij} gives the number of paths of length 2 connecting i and j.

It is known that $(A^2)_{ij}$ gives the number of paths of length 2 between node i and node j. But since $(i \to \alpha \to j) \leftrightarrow (j \to \alpha \to i)$, we have

$$\boxed{N = \frac{1}{2} A^2}$$

d.) We determine the total number of triangles in a network. A triangle can be thought of as a path of length 3 from a node, i, to itself. This is given by $(A^3)_{ii}$. But since all paths around



are to count as a single triangle, and there are 6 such paths, we conclude that the total number of triangles is given by

$$\boxed{\frac{1}{6} \operatorname{Tr} A^3}$$

6.51

a.) We show that a 3-regular graph must have an even number of nodes.

Let n be the number of nodes.

$$c = \frac{1}{n}\sum_{i=1}^{n} k_i = \frac{1}{n}\sum_{i=1}^{n} 3 = \frac{3}{n}\sum_{i=1}^{n} 1 =$$

$$\frac{3}{n}\cdot n = 3. \quad c = \frac{2m}{n} \Rightarrow 3 = \frac{2m}{n} \Rightarrow$$

$$n = 2\left(\frac{m}{3}\right) \in \mathbb{N}$$ since the graph is 3-regular.

Thus, the number of nodes of a 3-regular graph is even. $\boxed{\mathbb{Z}}$

b.) We show that the average degree of a tree is strictly less than 2.

The average degree is given by

$$c = \frac{2m}{n} = 2\cdot\frac{n-1}{n} = 2(1-1/n) < 2$$

since trees have always satisfy $m = n-1$. $\boxed{\mathbb{Z}}$

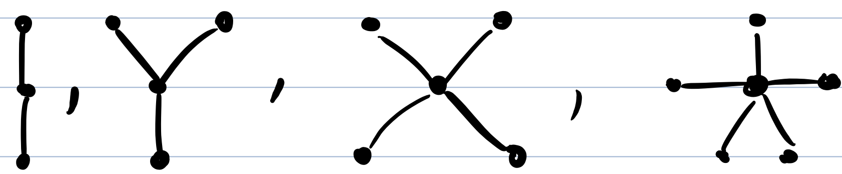
c.) We consider paths between nodes A, B, and C. The edge-connectivity (i.e. the number of edge-independent paths) between A and B is x, and the edge-connectivity of B and C is $y < x$.

We determine the edge connectivity between A and C. We propose that any edge-independent path from $A \rightarrow B$ can be combined with any edge-independent path from $B \rightarrow C$ to form an edge-independent path from $A \rightarrow C$.

Thus, the edge connectivity of A and C is $\boxed{x \cdot y.}$

6.61

A star graph consists of $n-1$ nodes connected to a central node.



WLOG, we call this node 1. Then, the adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & & & 0 \\ 1 & & 0 & & \vdots \\ \vdots & & & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad (n \times n)$$

We construct the pattern.

Case 1: $n=2$.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 = \lambda^2 - 1 \Rightarrow$$

$\lambda_{max} = 1.$

Case 2: $n=3$.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$= 2\lambda - \lambda^3 = -\lambda(\lambda^2 - 2) \Rightarrow \lambda_{max} = \sqrt{2}.$$

Case 3: $n=4$.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0 = \lambda^2(\lambda^2 - 3)$$

$$\Rightarrow \lambda_{max} = \sqrt{3}.$$

It is trivial to confirm computationally that the characteristic equation always takes the form

$$0 = \pm \lambda^{n-2}(\lambda^2 - (n-1)) \Rightarrow \boxed{\lambda_{max} = \sqrt{n-1}}$$

I either did not attempt or was unable to solve the other exercises from this chapter.