3.3.1

We consider an improved model for a laser:

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p$$

Where
- n is the number of laser photons
- N is the number of excited atoms
- G is the gain coefficient for the process of "stimulated emission"
- k is the decay rate due to loss of photons by transmission through the mirrors
- f is the decay rate for spontaneous emission
- p is the pump strength

$G, k, f > 0$.

We convert it to a 1D system for simpler analysis.

a.) To do so, we will use adiabatic elimination. That is, if we suppose N relaxes much faster than n, then we can approximate $\dot{N} \approx 0$. Using this approximation, we reduce this model of a laser into a first-order system:

$$\dot{N} = -GnN - fN + p \approx 0 \implies 0 = (-Gn - f)N + p$$

$$\Rightarrow N = \frac{p}{Gn+f} \Rightarrow$$

$$\boxed{\dot{n} = \frac{Gnp}{Gn+f} - kn}$$

b.)

We show that $n^* = 0$ becomes unstable for some $p > p_c$.

$$f(n) = \frac{Gnp}{Gn+f} - kn \Rightarrow f'(n) = \frac{1}{(Gn+f)^2}\left(Gp(Gn+f) - Gnp \cdot G\right) - k$$

$$\Rightarrow f'(0) = \frac{1}{f^2}\left(Gpf\right) - k = \frac{Gp}{f} - k > 0 \Rightarrow p > \frac{kf}{G} = p_c$$

That is, zero photons becomes unstable as soon as the pump strength is greater than kf/G.

c.) Transcritical.

3.3.2

We consider the Maxwell-Bloch equations, which form an even better model for a laser:

$$\dot{E} = K(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$

$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda E P)$$

where

- E is electricfield
- P is mean polarization
- D is population inversion
- κ decay rate in cavity due to beam transmission
- γ_1: decay rate of polarization
- γ_2: decay rate of population inversion

eh...