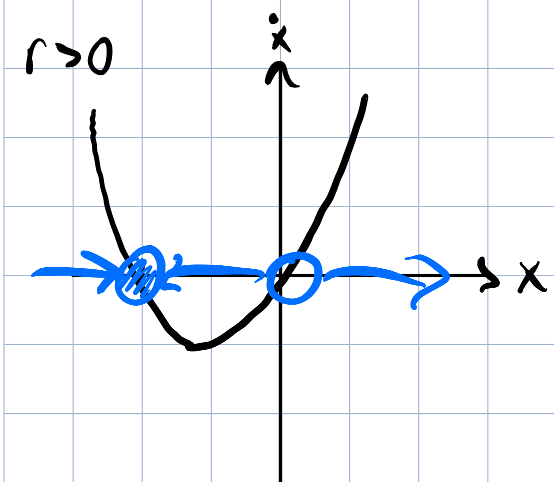
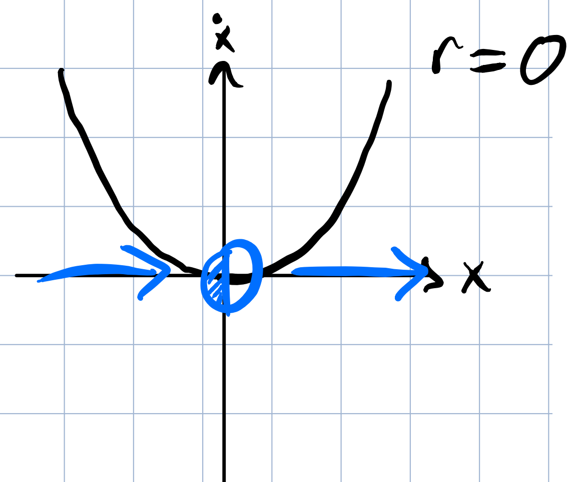
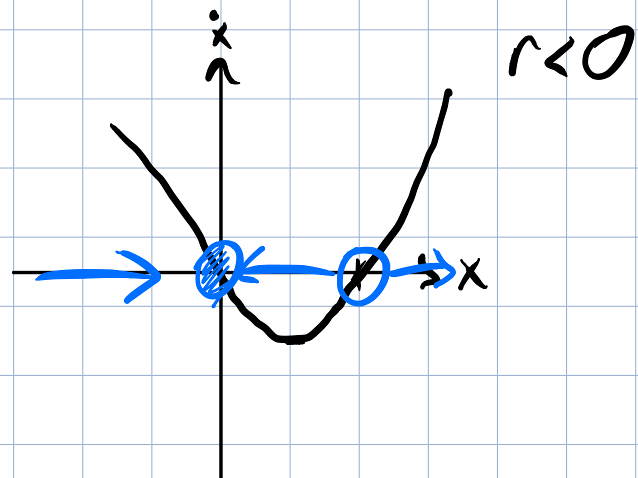
For the following systems, we sketch the system's phase portraits as we vary r. We show that a transcritical bifurcation occurs at some r. Then, we sketch the system's bifurcation diagram.

3.2.1

$$\dot{x} = rx + x^2 = x(r+x)$$







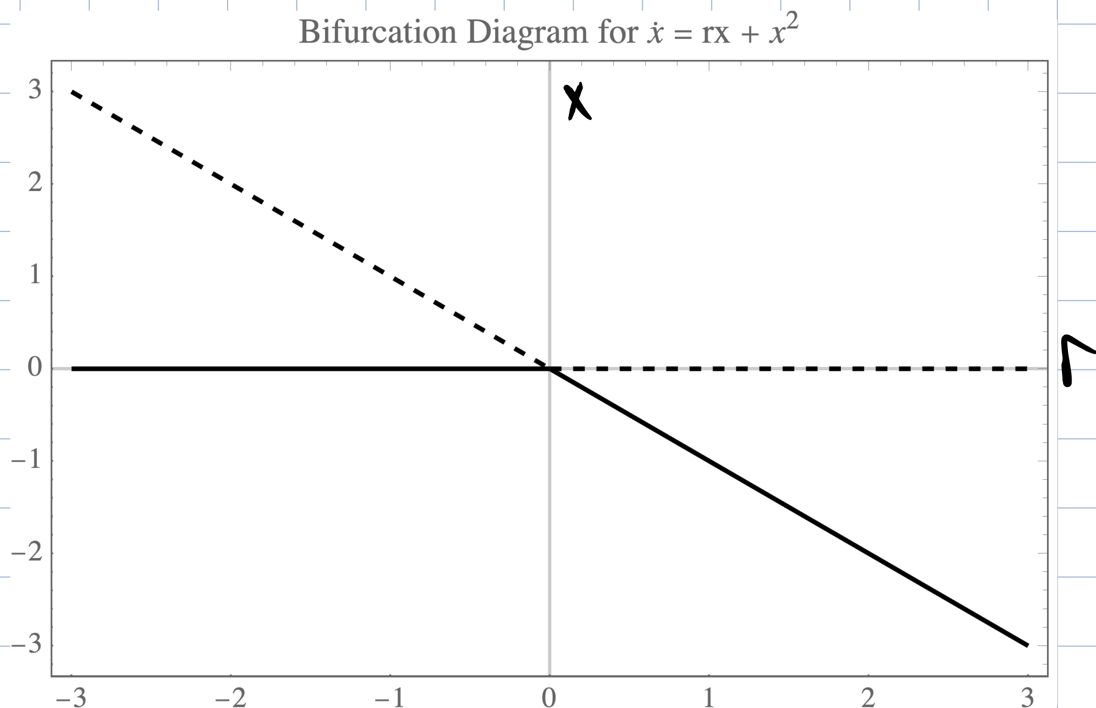
We observe that a transcritical bifurcation occurs at $x=0$, since its stability changes as we vary r. This transcritical bifurcation occurs when $r=0$.

The system's fixed points occur when $x(r+x)=0 \Rightarrow x^*=0, x^*=-r$

$f(x) = rx+x^2 \Rightarrow f'(x) = r+2x \Rightarrow f'(0) = r$, stable $r<0$, unstable $r>0$

$f'(-r) = r + 2(-r) = -r$ stable $r>0$
unstable $r<0$

Hence, our bifurcation diagram takes the form

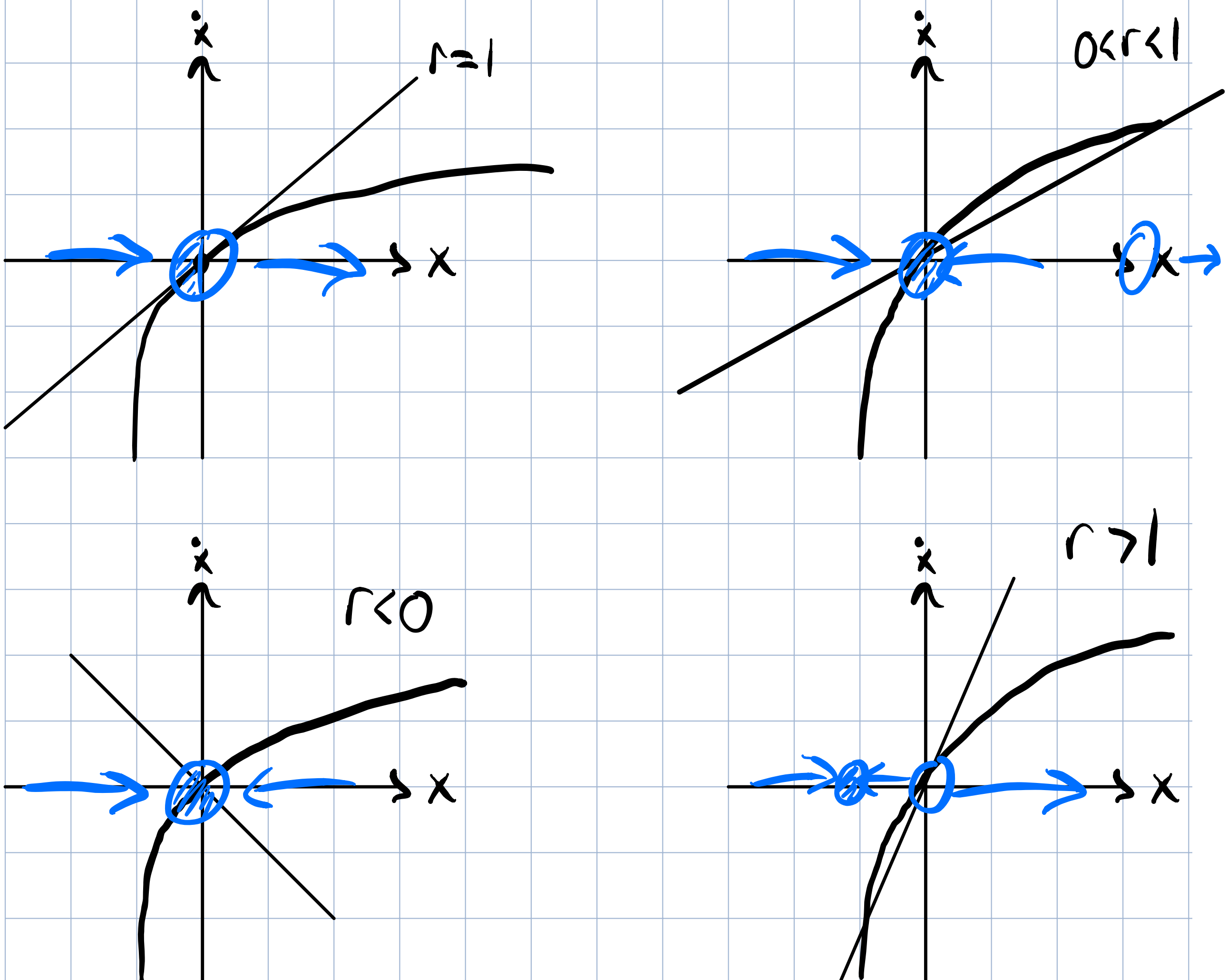


This recreates the qualitative behavior we see in our phase portraits: stable fixed point is $x^* = -r$ when $r>0$. As you decrease r, it makes a semistable fixed point at $x^* = 0$. Then, for $r<0$, $x^*=0$ is stable.

3.2.2

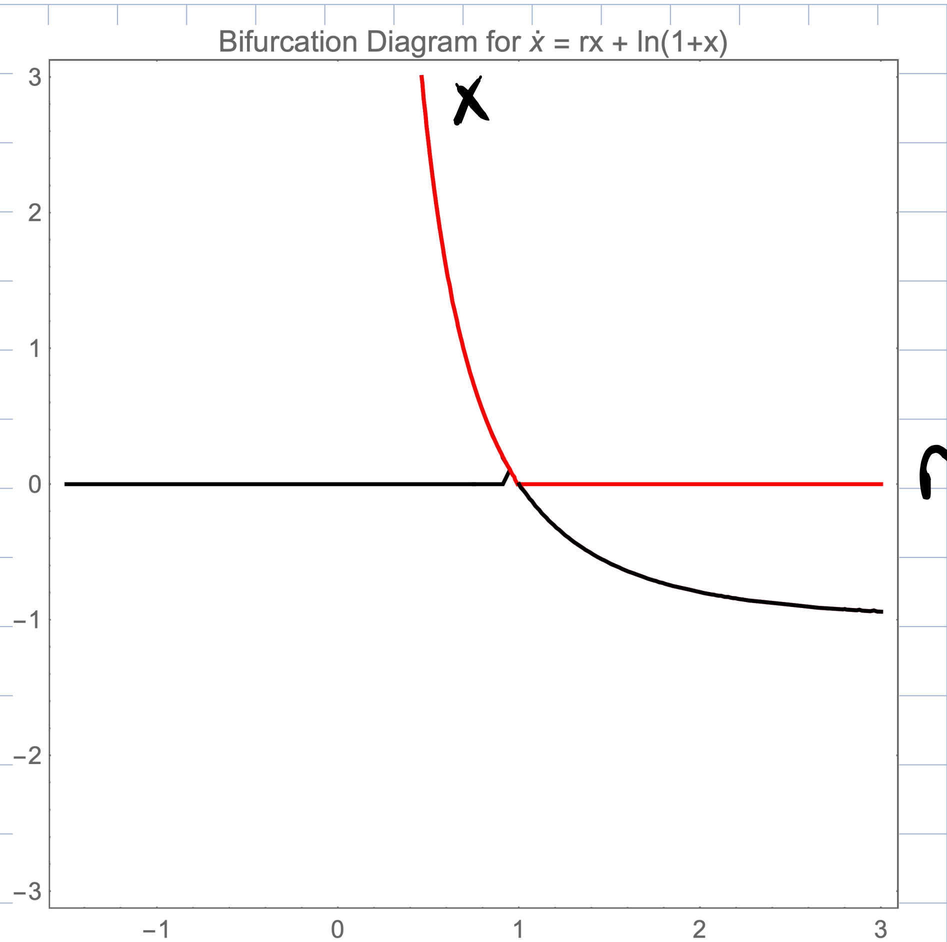
$\dot{x} = rx - \ln(1+x)$

We plot this system for various values of r below.



Although this system allows for more complex behavior than the previous one, there is still a transcritical bifurcation at $x=0$, which occurs when $r=1$. When $r<1$, this fixed point is stable. For $r>1$, it is unstable.

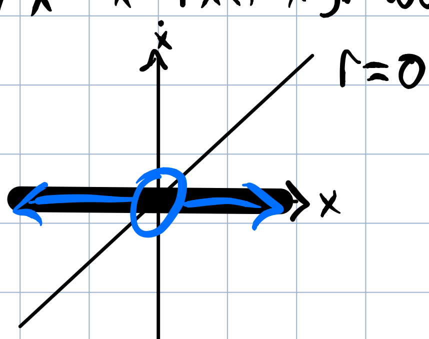
Accordingly, we have the following bifurcation diagram (red corresponds to unstable solution)

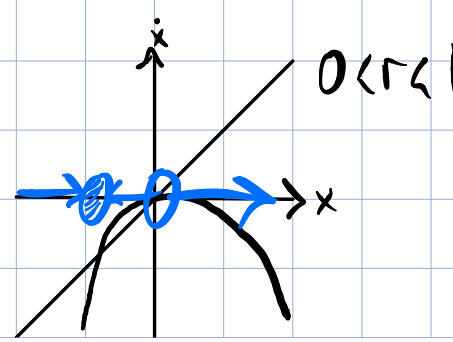


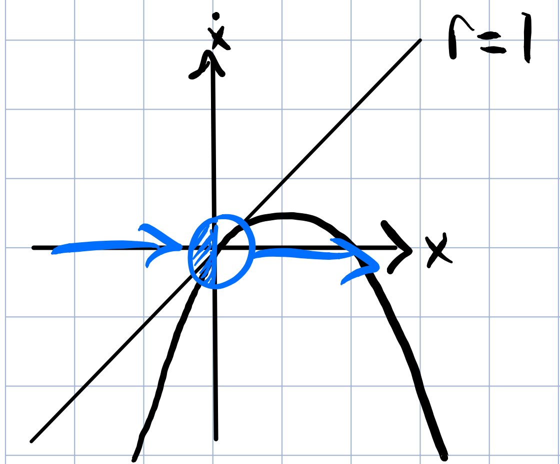
There is a small artifact at $(1,0)$. I need a new implicit equation plotter!

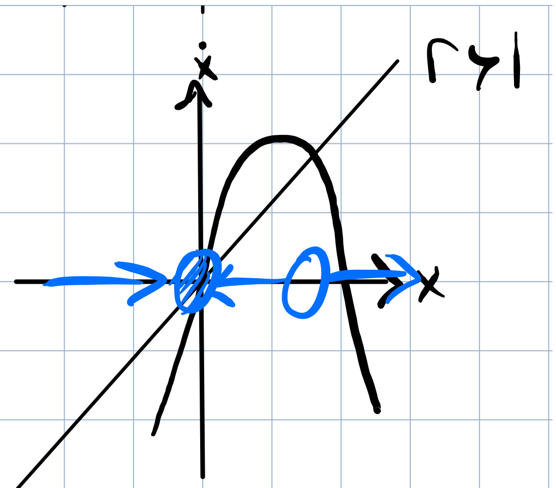
3.2.3

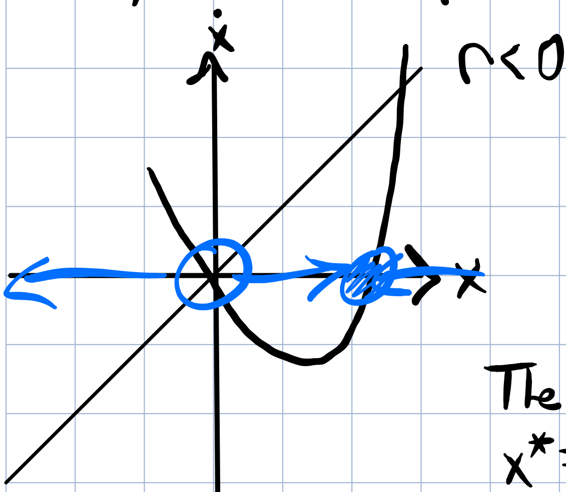
Let $\dot{x} = x - rx(1-x)$. We plot phase portraits for various values of r.











Clearly, we have a transcritical bifurcation when $x=0$, $r=1$.

The fixed points of this system are $x^*=0$, and the other solutions to

$$x = rx(1-x) \Rightarrow 1 = r(1-x) \Rightarrow$$

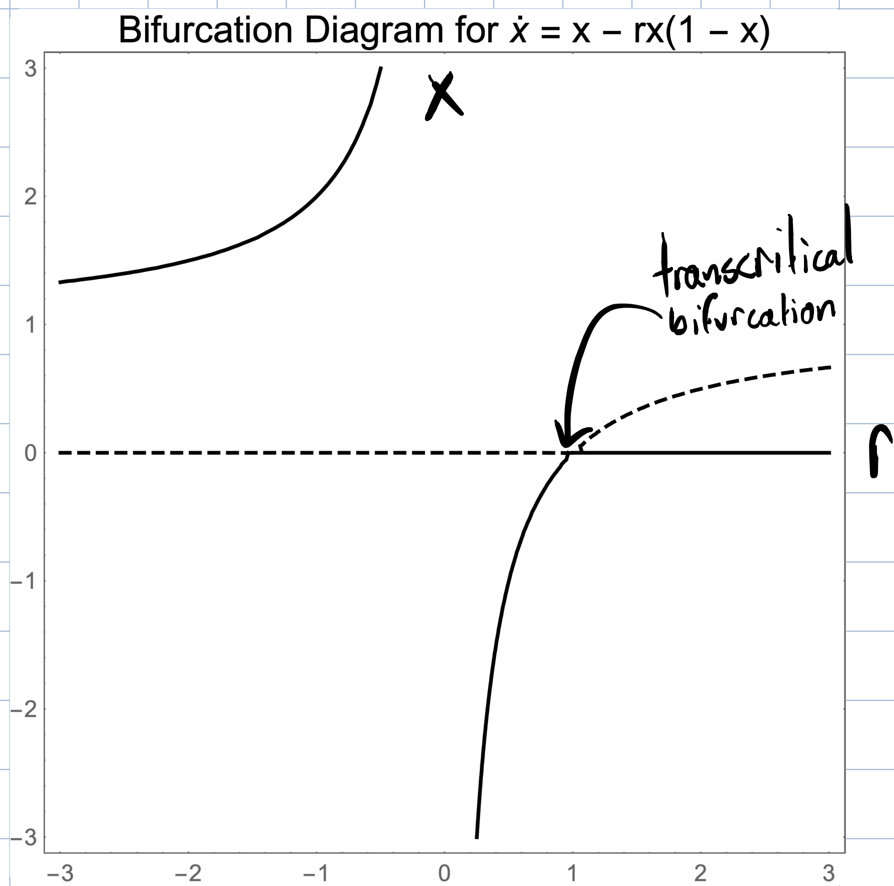
$$1 = r - rx \Rightarrow rx = r-1 \Rightarrow x^* = \frac{r-1}{r}$$

$$f(x) = x - rx + rx^2 \Rightarrow f'(x) = 1 - r + 2rx$$

$$f'\left(\frac{r-1}{r}\right) = 1 - r + 2r\frac{r-1}{r} = 1 - r + 2r - 2$$

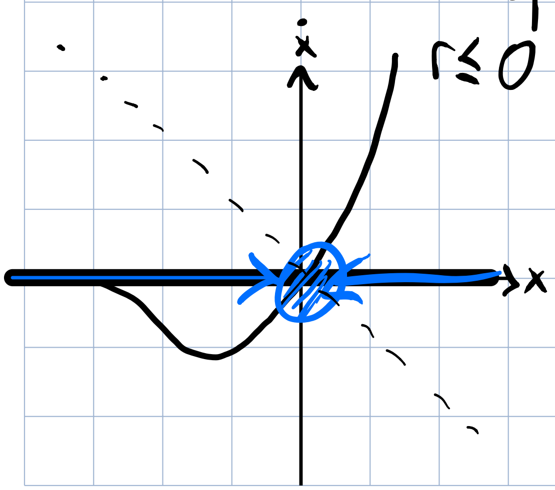
$$= r-1 \begin{cases} \text{stable if } r<1 \\ \text{unstable if } r \geq 1 \end{cases}$$

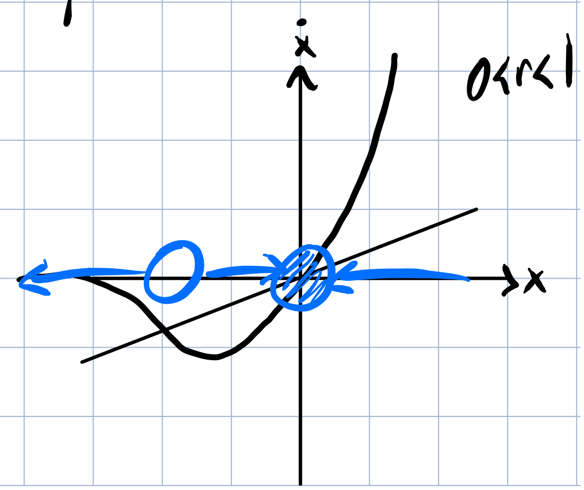
We thus have the following bifurcation diagram:

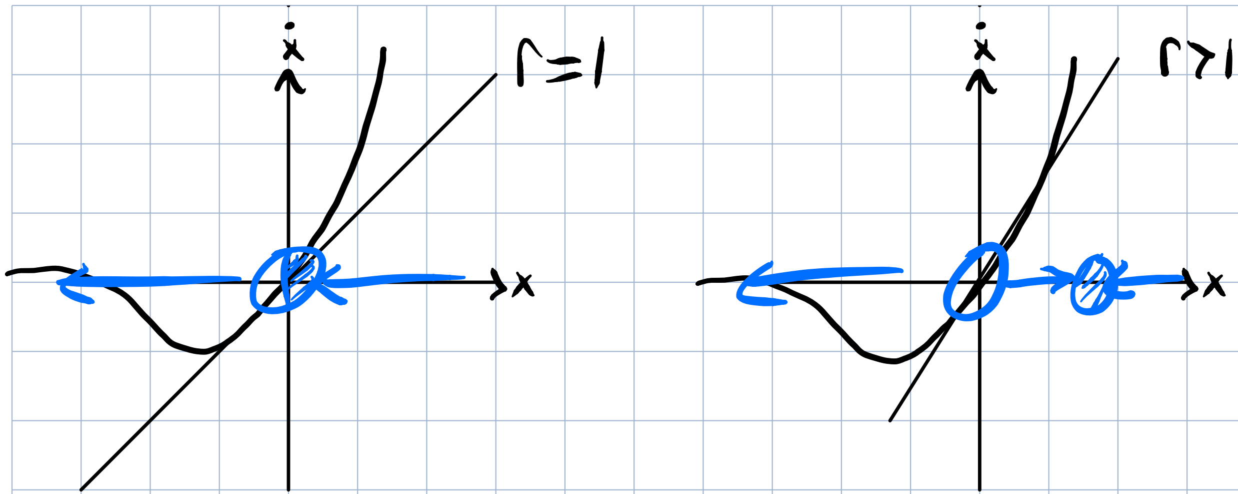


3.2.4

Let $\dot{x} = x(r - e^x)$. We plot the phase portraits for various r.







There is a transcritical bifurcation when $r=1$. This is clearly illustrated in the bifurcation diagram: