For the following systems, we sketch the qualitatively different vector fields that occur as we vary r. We show that a saddle-node bifurcation occurs at some r_c, and we find that r_c. We finally sketch the bifurcation diagram.

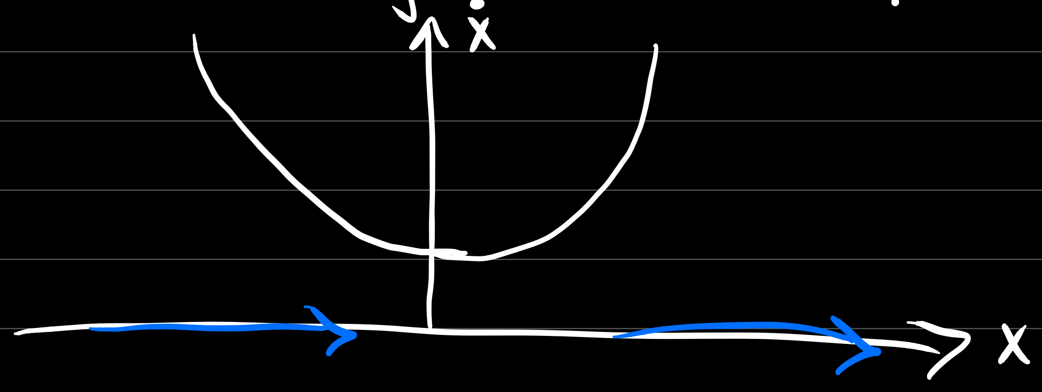
3.1.1

$\dot{x} = 1 + rx + x^2$

This system has fixed points when

$$1 + rx + x^2 = 0$$

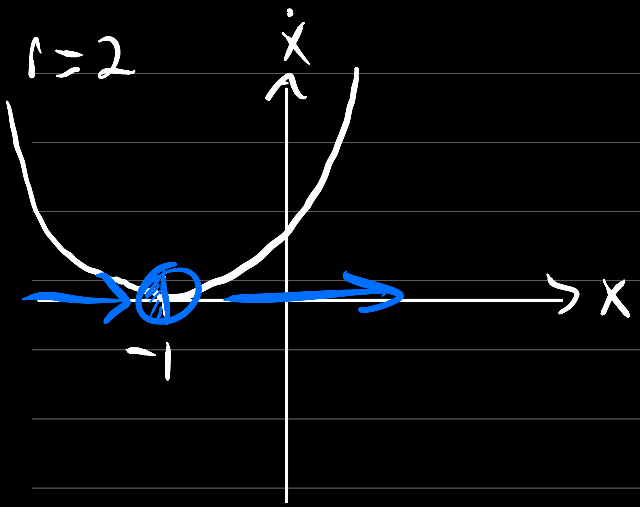
Clearly, when $r = 0$, the system has no fixed points.

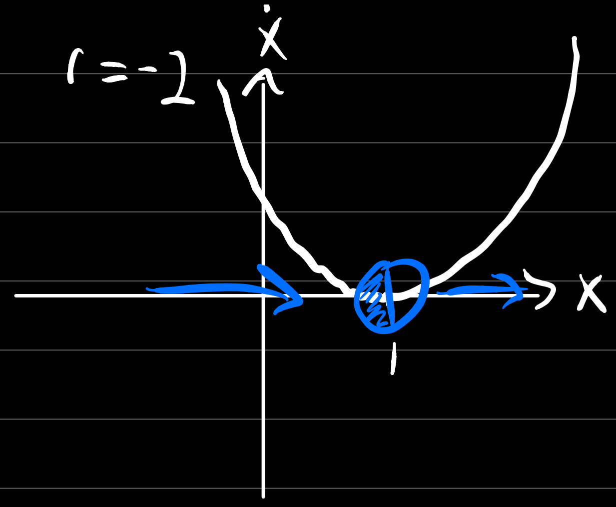


When $r = \pm 2$, $1 \pm 2x + x^2 = (x \pm 1)^2 = 0$

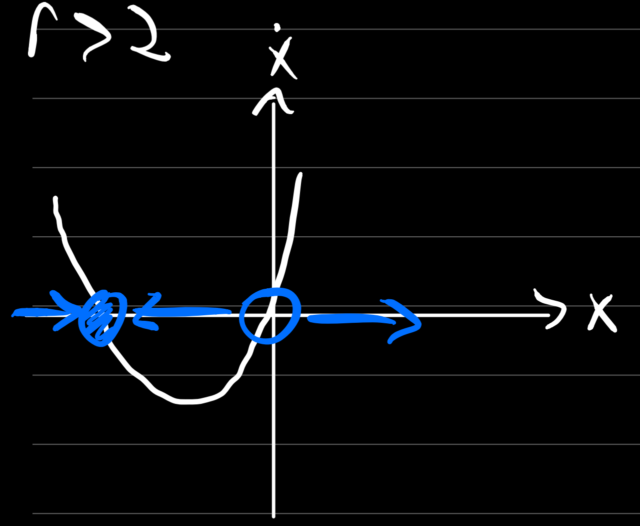
$\Rightarrow x = \mp 1$

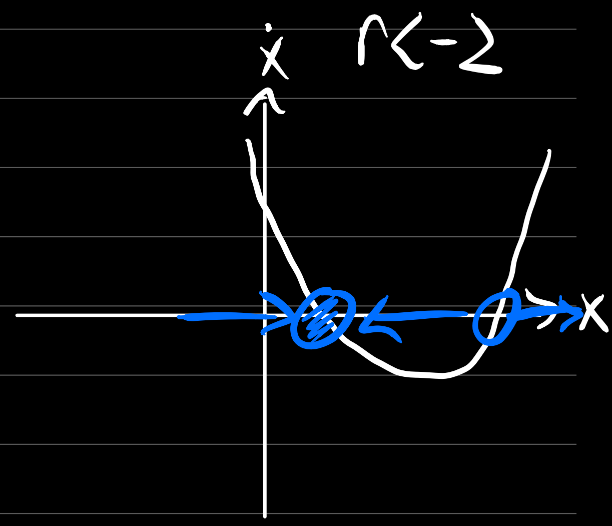
And the system's phase diagram is as follows





When $|r| > 2$, we have 2 fixed points





So it appears that this system has two saddle-node bifurcations at $r_c = \pm 2$.

We find the locations of these fixed points explicitly

$$1 + rx + x^2 = 0 \Rightarrow x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

$$= \frac{-r}{2} \pm \sqrt{\frac{1}{4}(r^2-4)} = \frac{-r}{2} \pm \sqrt{\frac{r^2}{4}-1}$$

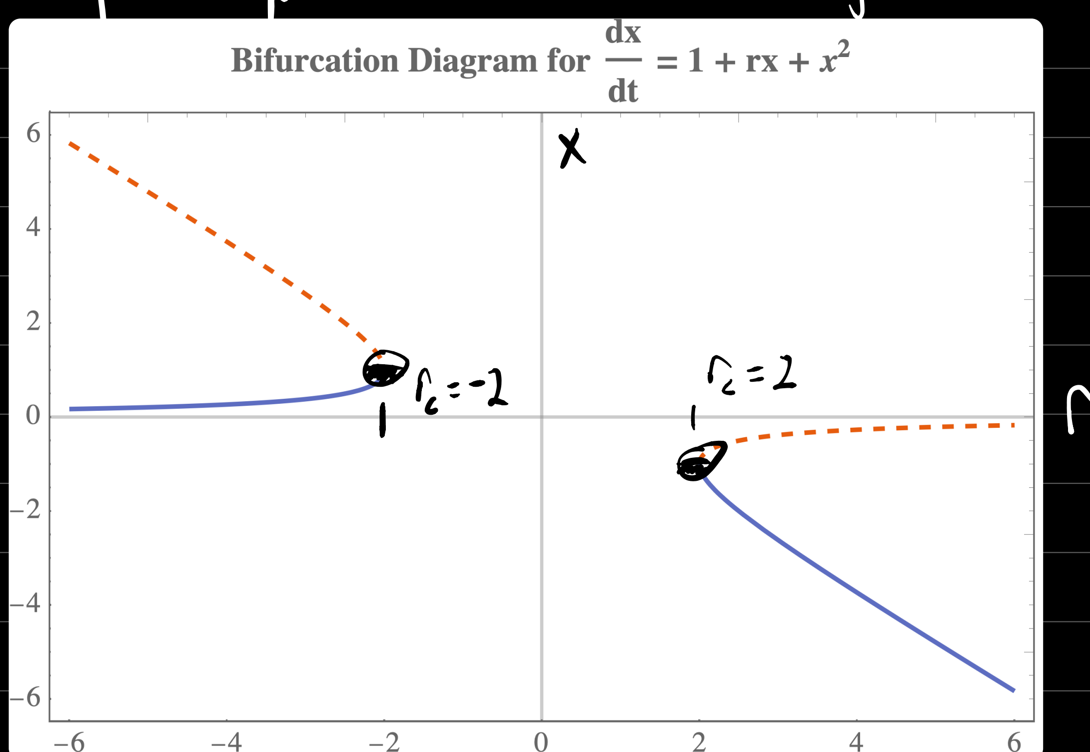
Which fixed point is stable? Using linear stability analysis, we simply check the sign of $f'(x^*)$.

$$f' = r+2x \Rightarrow f'(x^*) = r + 2\left(\frac{-r}{2} \pm \sqrt{\frac{r^2}{4}-1}\right)$$

$$= r - r \pm \sqrt{\frac{r^2}{4}-1} = \pm\sqrt{\frac{r^2}{4}-1}.$$

So, we confirm analytically that when $|r|>2$, we have a stable fixed point at $-\sqrt{r^2/4-1}$, and an unstable fixed point at $\sqrt{r^2/4-1}$.

We may now plot the bifurcation diagram.



We obtained this graph by simply plotting

$$x(r) = \frac{-r}{2} \pm \sqrt{\frac{r^2}{4} - 1}$$

and using the fact that the negative root is stable.

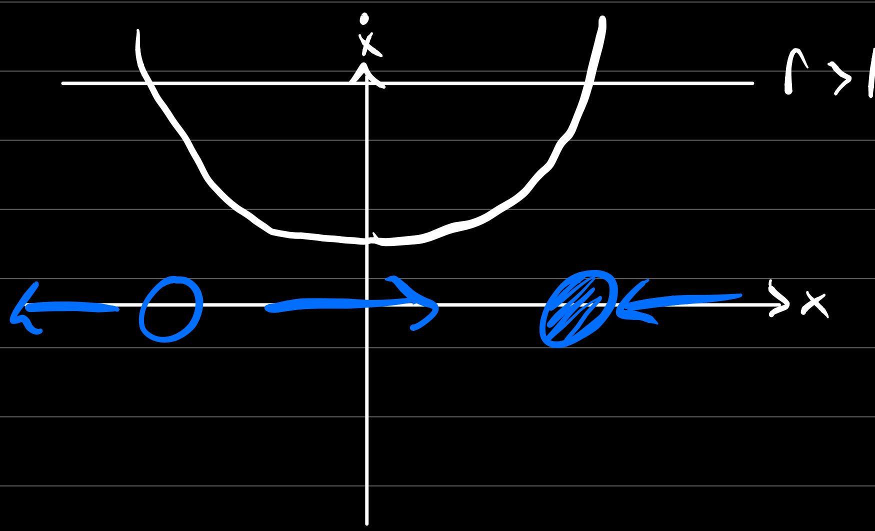
3.1.2

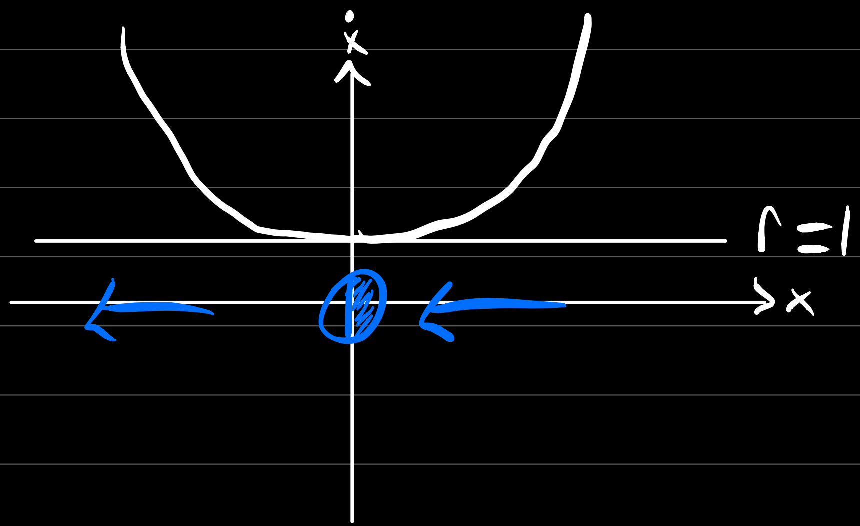
Let $\dot{x} = r - \cosh x$. To see the fixed points, we plot

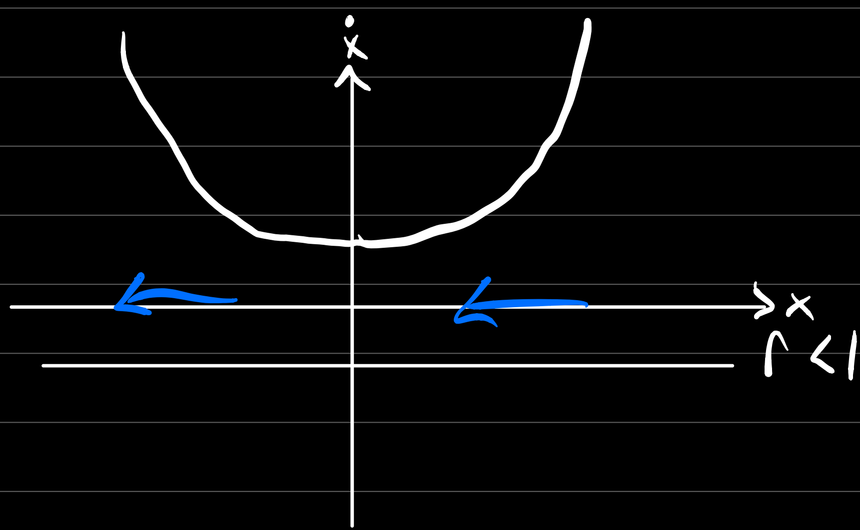
$\dot{x} = r$

$\dot{x} = \cosh x$

and we recall that flow is to the right when $r > \cosh x$, to the left when $r < \cosh x$, and fixed points occur when $r = \cosh x$.







Clearly, a saddle node bifurcation occurs when $r_c = 1$. The fixed points occur when $r = \cosh x \implies x = \pm \cosh^{-1}(r)$.

We use linear stability analysis:

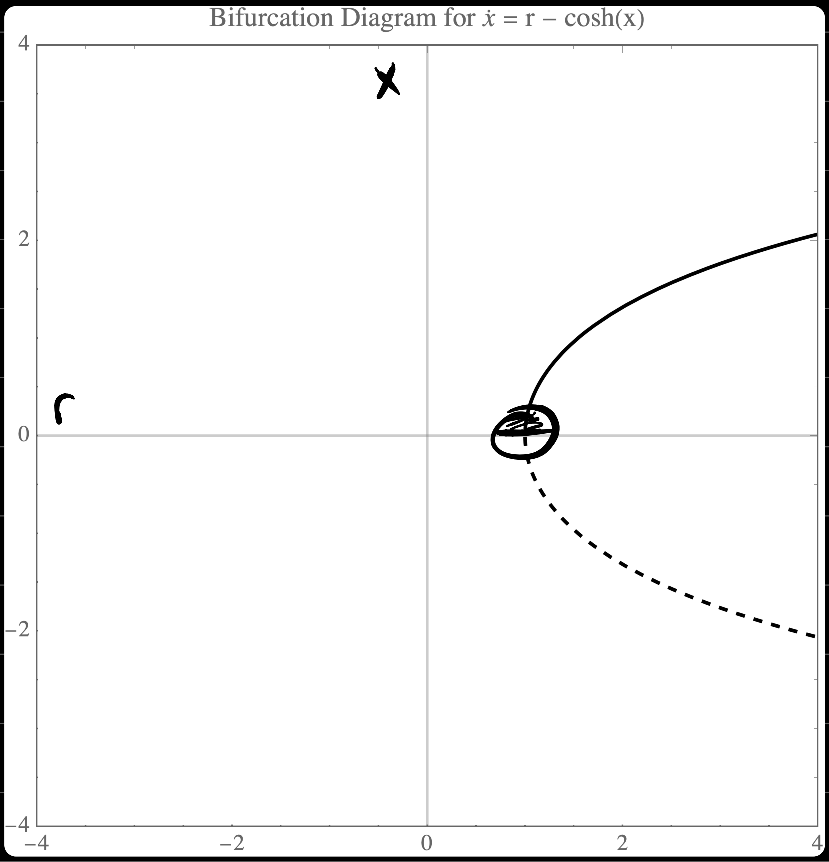
$$f' = -\sinh x \Rightarrow f'(\cosh^{-1}(r)) =$$

$$-\sinh(\cosh^{-1}(r)) = -\sqrt{(r+1)(r-1)} = -\sqrt{r^2-1} < 0$$

$$f'(-\cosh^{-1}(r)) = \sqrt{(r-1)(r+1)} = \sqrt{r^2-1} > 0$$

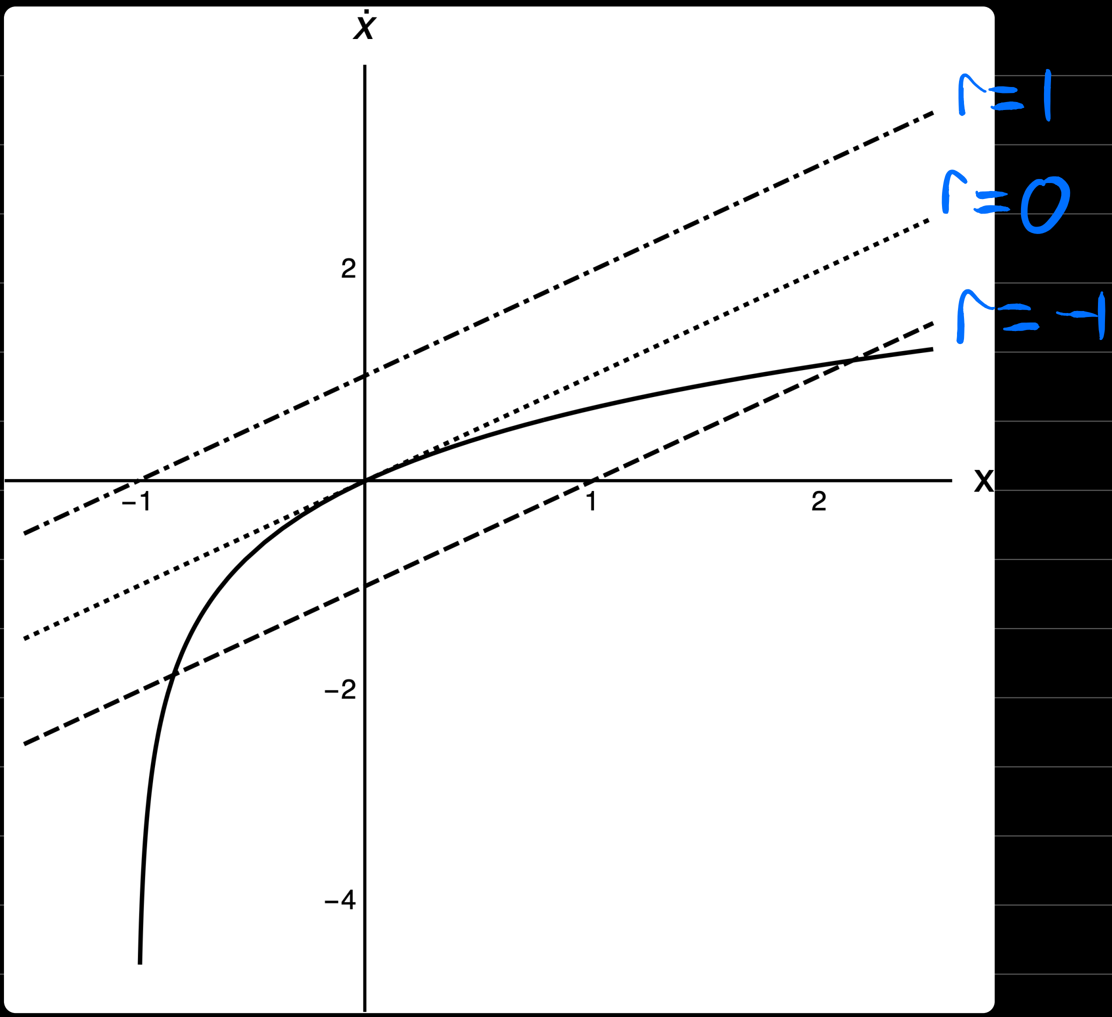
So, the positive root of $r - \cosh x = 0$ is stable. The negative is unstable.

Accordingly, our bifurcation diagram is as follows

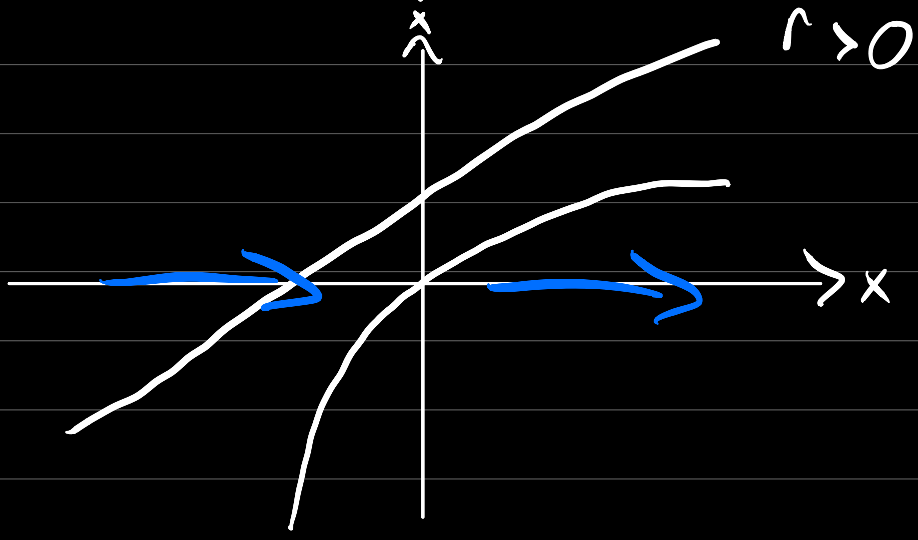


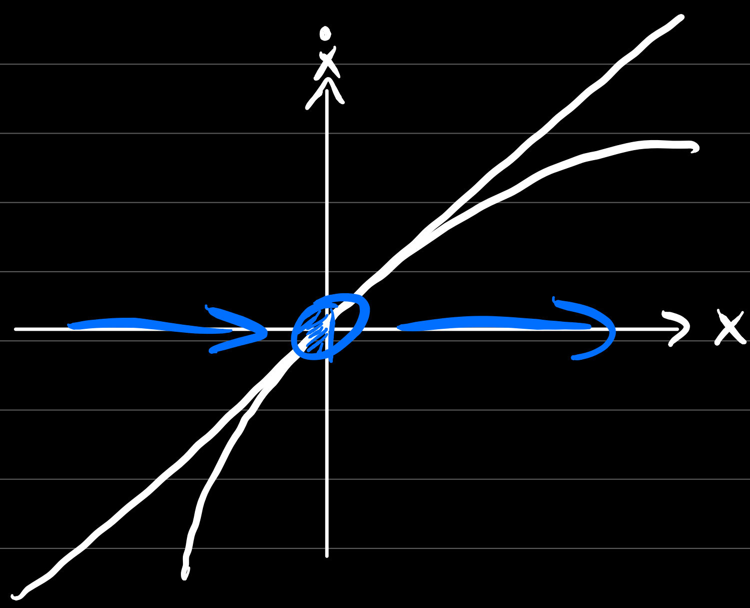
3.1.3

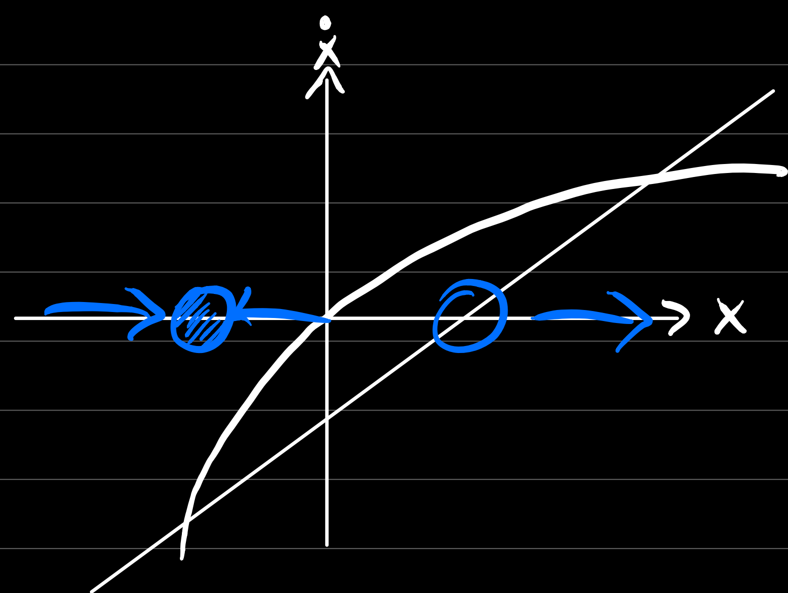
$\dot{x} = r + x - \ln(1+x)$. To see the behavior of fixed points of this system, we plot $\dot{x} = r + x$, $\dot{x} = \ln(1+x)$. Flow to right when $r + x > \ln(1+x)$.



Clearly, the system has one bifurcation. It occurs when $r_c = 0$. The associated vectorfields are





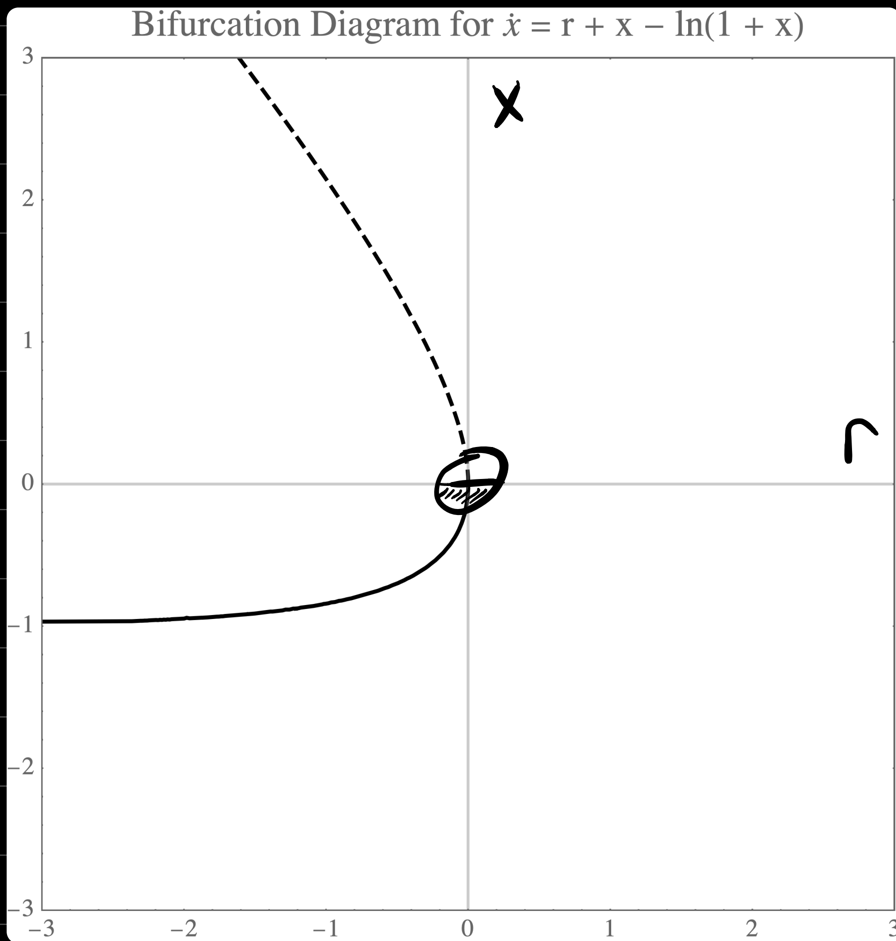


The fixed points occur when

$$r + x = \ln(1+x)$$

which does not have an analytic solution. However, the fixed point that corresponds to $x < 0$ is stable and the other is unstable.

We plot the bifurcation diagram below

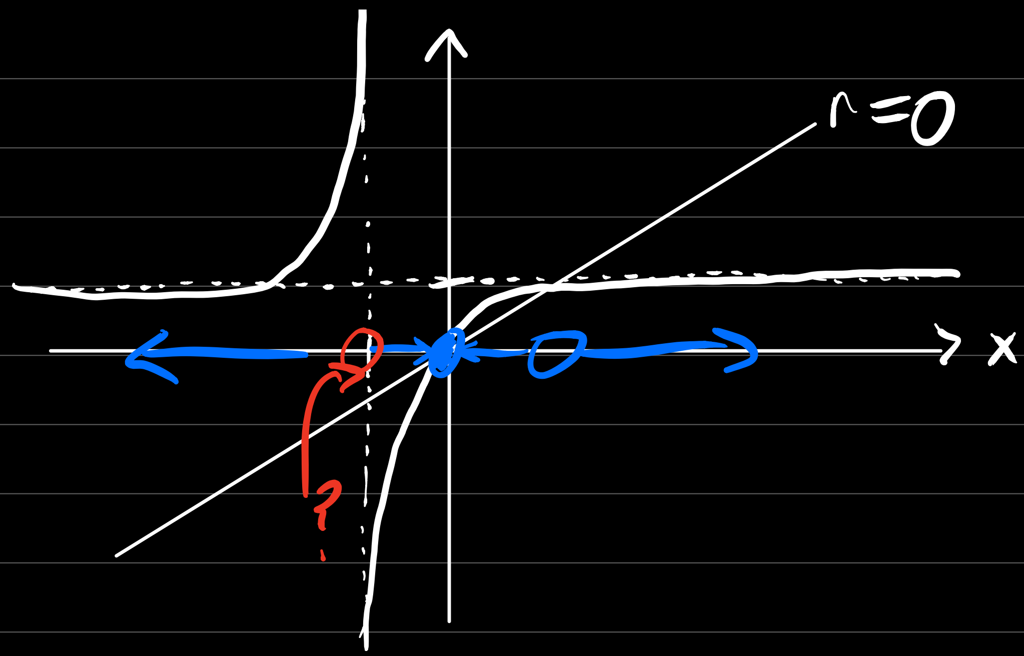


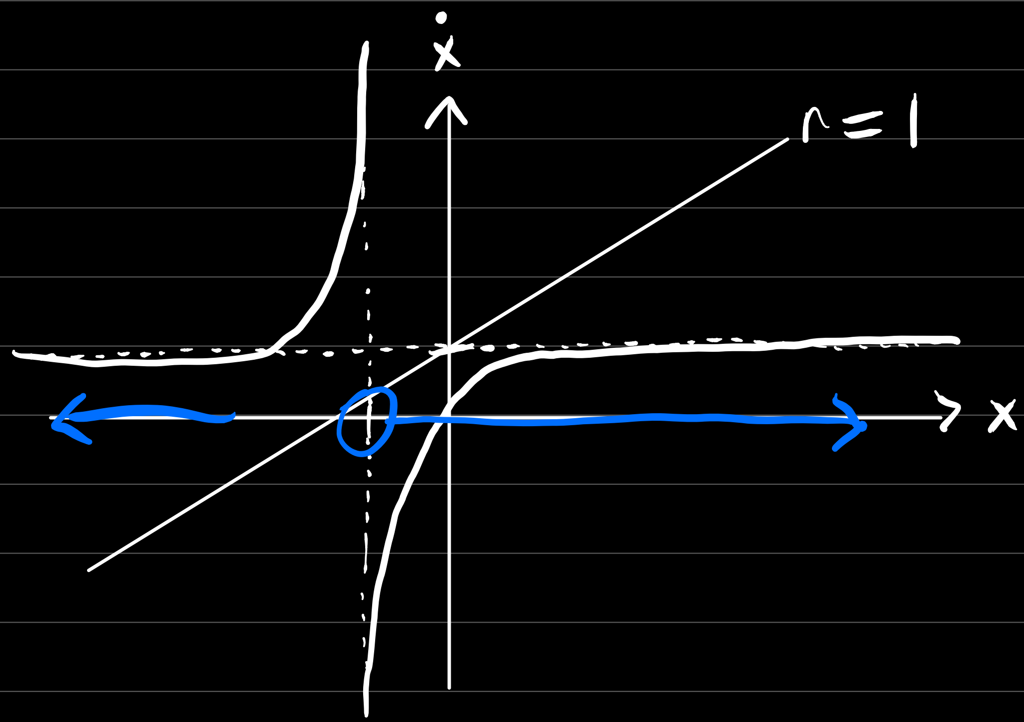
3.1.4

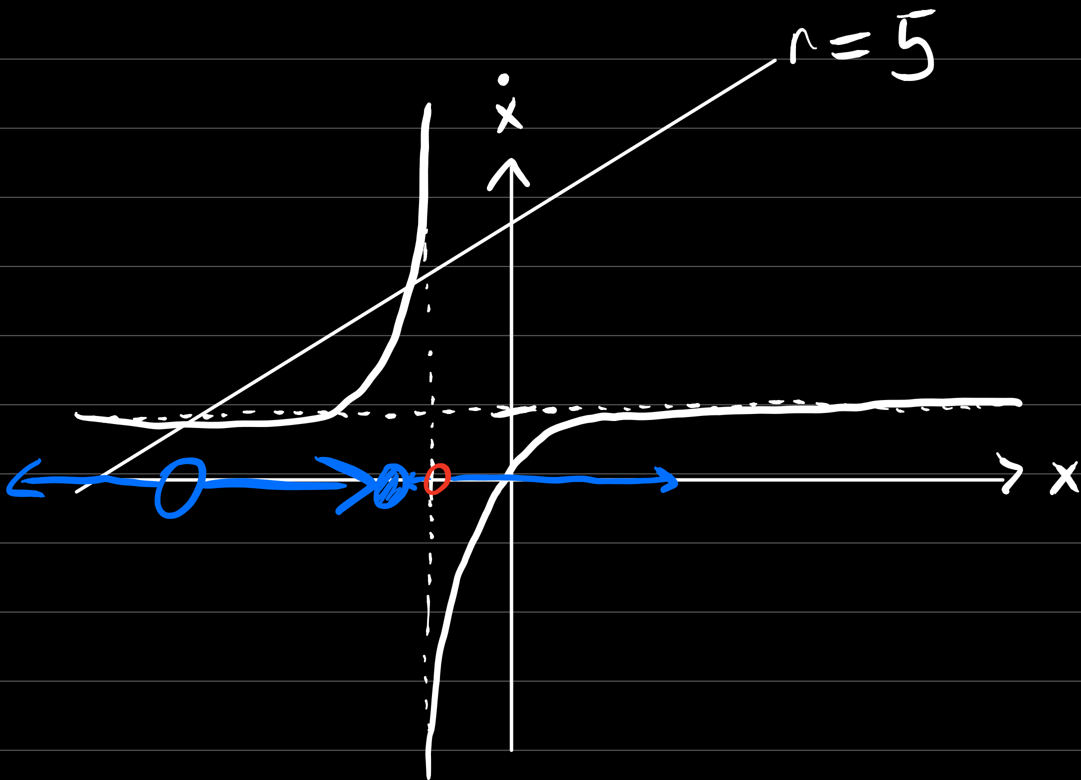
Let $\dot{x} = r + x/2 - x/(1+x)$. We plot $r + x/2$ and $\frac{x}{1+x}$.

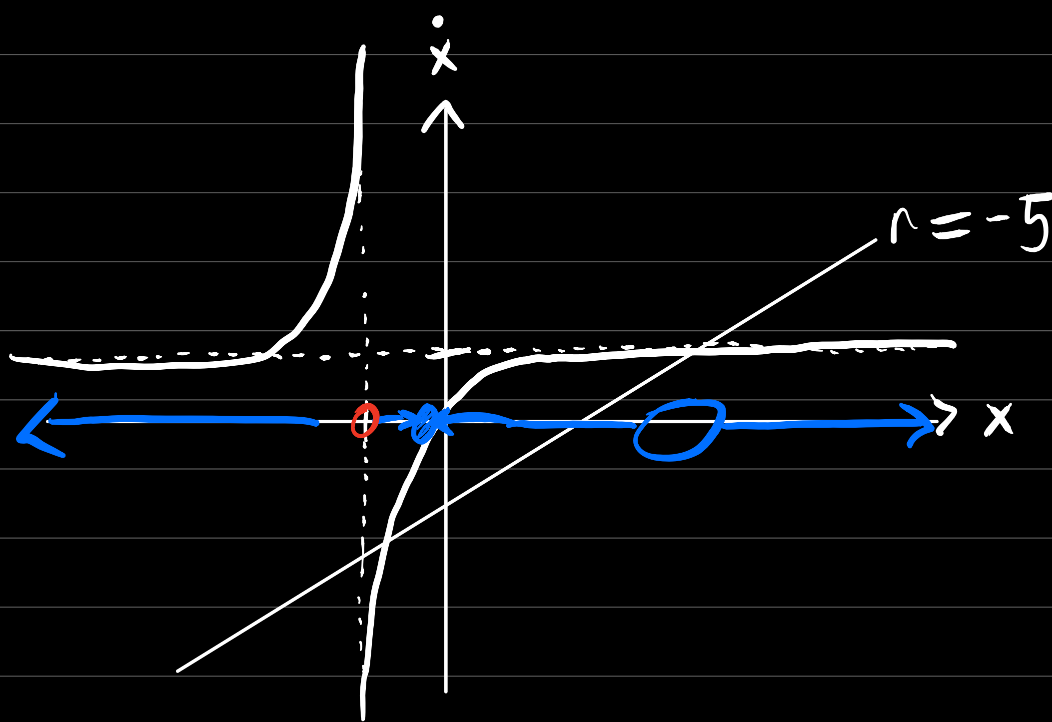
Flow is to the right when $r + x/2 > x/(1+x)$

$\dot{x}$


r=0
x
?


ẋ
r=1
x





This system appears to either have one fixed point, or three.
(However, the point where $x/1+x$ diverges is complicated since the functions

aren't equal. Their difference does in fact change sign though.

The divergence occurs at $x^* = -1$. The others occur when

$$r + x/2 - x/(1+x) = 0 \Rightarrow$$

$$(r + x/2)(1+x) = x \Rightarrow r + rx + \frac{x}{2} + \frac{x^2}{2} - x = 0$$

$$\Leftrightarrow \frac{x^2}{2} + x\left(r - \frac{1}{2}\right) + r = 0. \Rightarrow$$

$$x = \left(\frac{1}{2} - r\right) \pm \sqrt{\left(r - \frac{1}{2}\right)^2 - 4 \cdot \frac{1}{2} \cdot r} =$$

$$\frac{1}{2} - r \pm \sqrt{r^2 - r + \frac{1}{4} - 2r} =$$

$$\frac{1}{2} - r \pm \sqrt{r^2 - 3r + 1/4}.$$ So, we only have

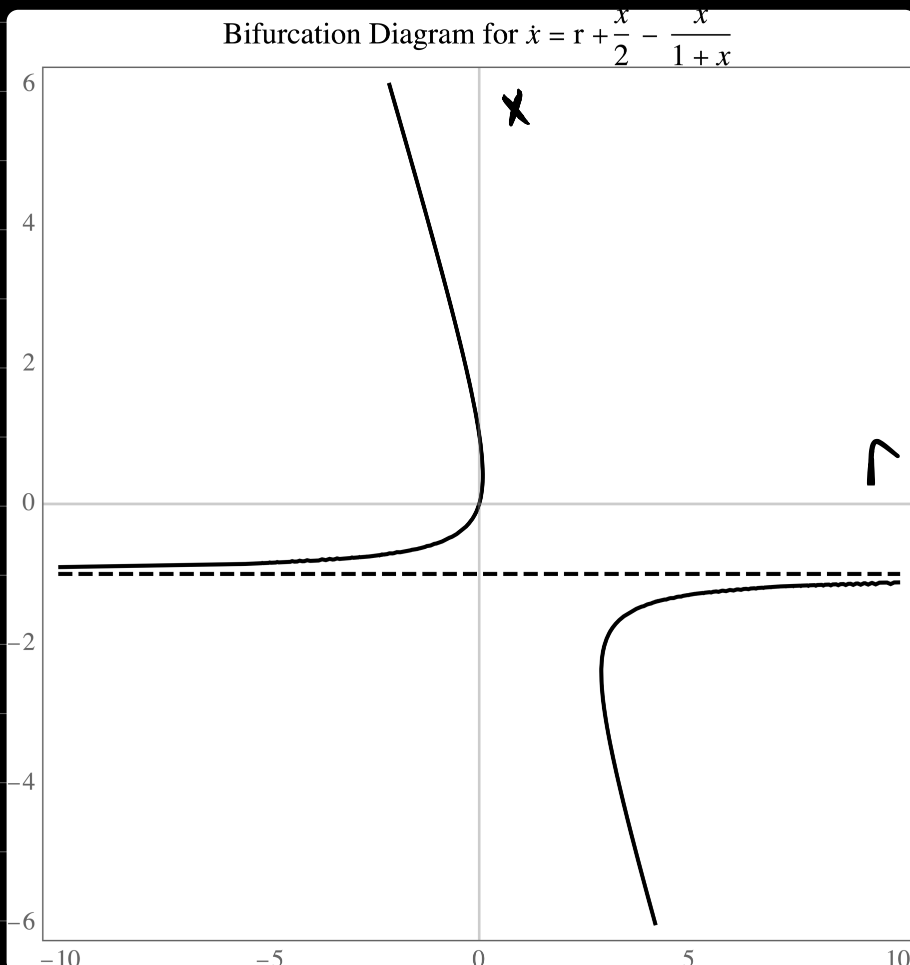
3 solutions when $r^2 - 3r + 1/4 \geq 0$.

We now use linear stability analysis on these fixed points:

$$f = r + x/2 - x/(1+x) \Rightarrow f' = \frac{1}{2} - \frac{1}{(1+x)^2}(1-x)$$

$$f'(-1) = \frac{1}{2} - \frac{1-(-1)}{(1+(-1))^2} = -\infty$$

This analysis is inconclusive. Generally speaking, without linearity stability, the bifurcation diagram is

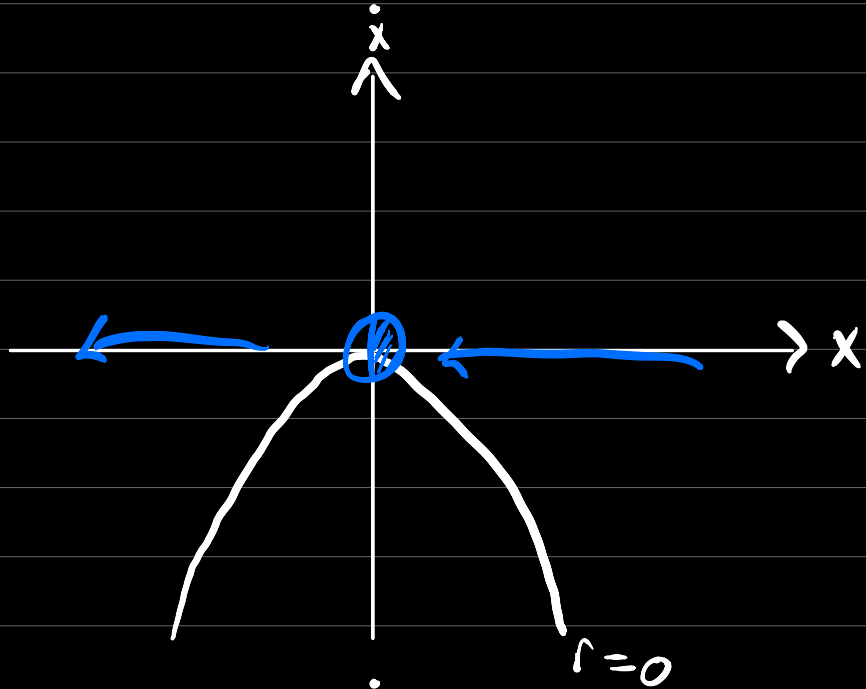


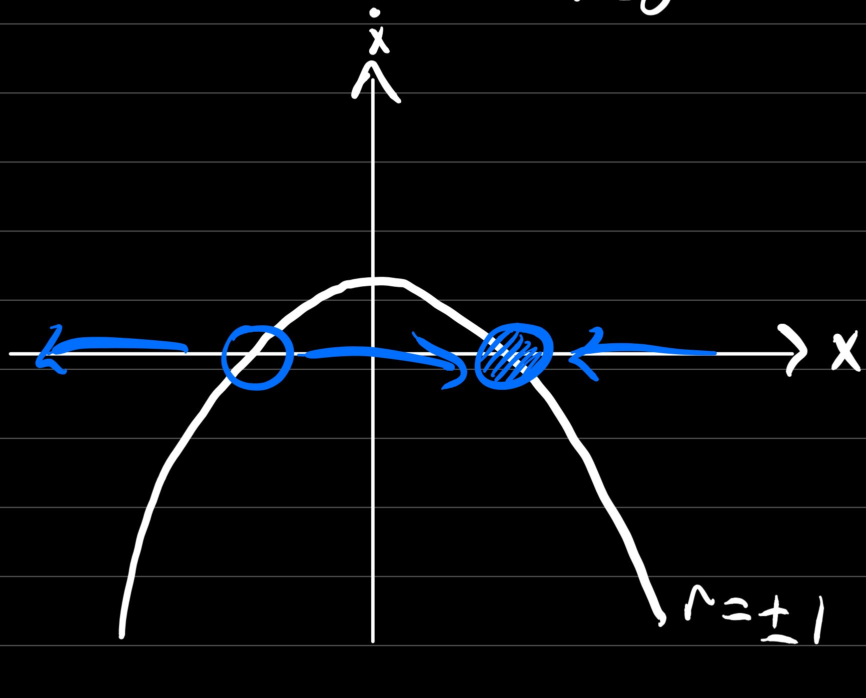
The diagram suggests that $x = -1$ is not a fixed point. But I do not know.

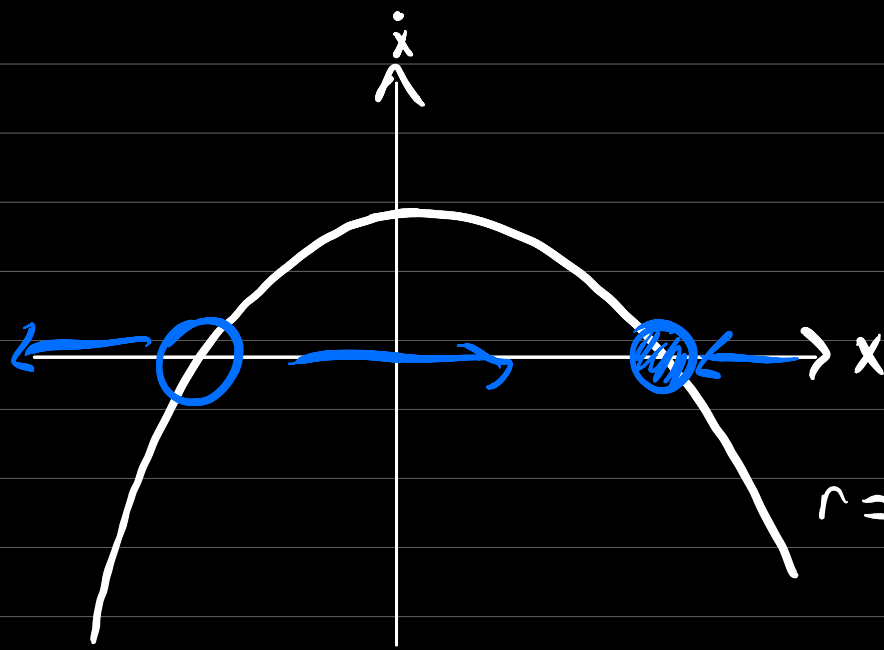
3.1.5

Let $\dot{x} = r^2 - x^2$. We sketch the vector fields and make a bifurcation diagram

a)





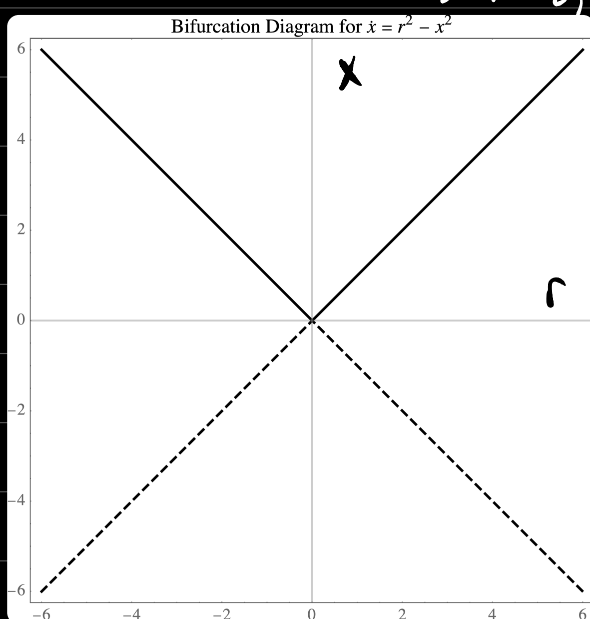


$r = \pm\sqrt{2}$

The system has two fixed points, except when $r=0$. Then, it has one. The fixed points occur when

$$r^2 = x^2 \implies x = \pm r$$

Qualitatively, we see that $x=r$ is stable, $x=-r$ unstable.



The bifurcation diagram clearly captures this behavior.

b.) Now, let $\dot{x} = r^2 + x^2$. Clearly, this system will only have a fixed point when $r=0$. It will be semistable