2.6.1

The text states that one dimensional systems cannot oscillate. Clearly, the system $m\ddot{x} = -kx$ oscillates, and describes the motion of a particle in one dimension.

The resolution of this apparent paradox comes from the fact that this system doesn't meet our definition of one dimensional:

$$\dot{q} = f(q)$$

Let $x = x_1$, $x_2 = \dot{x}_1$. Then, we have the system

$$\begin{cases} \dot{x}_2 = \frac{-k}{m} x_1 \\ \dot{x}_1 = x_2 \end{cases}$$

So, we implicitly had a two-dimensional system.

2.6.2

We prove that $\dot{x} = f(x)$ cannot oscillate by contradiction.

Let $0 < T$. Suppose $x(t) = x(t+T)$, and $x(t) \neq x(t+s)$ $\forall s \in (0,T)$.

We then consider

$$\int_t^{t+T} f(x)\,\frac{dx}{dt}\,dt = f(x(t))\,x(t)\Big|_t^{t+T} -$$

$$\int_t^{t+T} \frac{df}{dt}\cdot x(t)\,dt = f(x(t+T))\,x(t+T) - f(x(t))\,x(t)$$

$$-\int_t^{t+T} \frac{df}{dx}\,\frac{dx}{dt}\cdot x\,dt = f(x(t))\,x(t) - f(x(t))\,x(t)$$

$$-\int_t^{t+T} \frac{df}{dx}\,\frac{dx}{dt}\,x\,dt = -\int_t^{t+T} \frac{df}{dx}\,\frac{dx}{dt}\,x\,dt \Rightarrow$$

$$0 = \int_t^{t+T} \frac{df}{dx}\,\frac{dx}{dt}\,x\,dt + \int_t^{t+T} f\cdot\frac{dx}{dt}\,dt =$$

$$\int_t^{t+T} \frac{dx}{dt}\left(f(x) + x\cdot\frac{df}{dx}\right)dt = 0$$

We have two cases.

Case 1: $\dot{x} = 0 \Rightarrow x(t) = \text{const}$. This means that

$x(t) = x(t+s)$ for all $s \in (0, T)$, which is

a contradiction.

Case 2:

$$f(x) + x \cdot \frac{df}{dx} = 0 \Leftrightarrow \frac{df}{dx} = \frac{-f(x)}{x} \Rightarrow$$

$$-\int \frac{df}{f} = \int \frac{dx}{x} \Rightarrow -\ln f = \ln x + C_1 \Rightarrow$$

$$\ln f = -\ln x + C_2 \Rightarrow f = e^{-\ln x + C_2} = -Cx.$$

However,

$f(x) = \dot{x}$, so

$$\frac{dx}{dt} = -Cx \Rightarrow \int \frac{dx}{x} = \int -C\,dt$$

$\ln x = -Ct + D_1 \Rightarrow x(t) = De^{-ct}$. However, this violates our assumption that $x(t+T) = x(t)$.

Hence, we conclude that $x(t)$ can't be periodic. $\boxtimes$