2.5.11

We explore the case where a system reaches a fixed point in finite time.

Let $\dot{x} = -x^c$, $c \in \mathbb{R}$, $x \geq 0$.

a.) We find all c such that $x = 0$ is a stable fixed point.

We first observe that $x = 0$ is a fixed point $\forall c > 0$.

By linear stability analysis, a fixed point x^* is stable provided $f'(x^*) < 0$.

$$f'(x) = -cx^{c-1}$$

Thus, the only case where we can use linear stability analysis is when $c = 1$. This is stable.

More generally, the criterion for being a fixed point when $x \geq 0$ is that $f(x) < 0$ for x immediately greater than 0. This holds for all $c > 0$. Hence, we simply require

$$\boxed{c > 0}.$$

b.) Choose c such that $x^* = 0$ is stable. Can the particle ever reach the origin in finite time? Concretely, let $x(0) = 1$.

We find the solution analytically:

$$t+D_1 = \frac{dx}{-x^c} = \frac{x^{1-c}}{c-1} \Rightarrow (t+D_1)(c-1) = x^{1-c}$$

$$\Rightarrow x(t) = \left((c-1)(t+D_1)\right)^{\frac{1}{1-c}} = \left((c-1)t+D\right)^{\frac{1}{1-c}}.$$

$$x(0) = 1 = D^{\frac{1}{1-c}} \Rightarrow D = 1 \Rightarrow$$

$$x(t) = \left((c-1)t+1\right)^{\frac{1}{1-c}}$$

We just want to know if it is possible for the particle to arrive at the origin. Let $c<1$. Then

$$\frac{1}{1-c} > 1.$$

$$0 = \left((c-1)t+1\right)^{1/1-c} \Rightarrow 0 = (c-1)t+1 \Rightarrow$$

$$t = \frac{-1}{c-1}, \text{ where } 0 < \frac{-1}{c-1} < \infty.$$

We conclude that it is indeed possible for the particle to reach its fixed point in finite time.

2.5.2

We show that the solution to $\dot{x} = 1 + x^{10}$ diverges in finite time from any IC.

We recall that $\dot{x} = 1 + x^2$ has the solution

$x(t) = \tan(t + C)$, which diverges when $t + C = \pi/2$.

So long as $x > 1$, $1 + x^{10} > 1 + x^2$. Hence, when $x > 1$, the solution from $\dot{x} = 1 + x^{10}$ grows faster than the solution to $\dot{x} = 1 + x^2$.

$1 = \tan(t + C) \Rightarrow t = \frac{\pi}{4} - C$. So when

$t > \frac{\pi}{4} - C$, we know that $\dot{x} = 1 + x^{10}$ grows faster than $\dot{x} = 1 + x^2$.

Since the smaller diverges, the larger must as well.

2.5.3

Show that $\dot{x} = rx + x^3$ diverges in finite time for $r > 0$, $x_0 \neq 0$.

$$t + C_1 = \int \frac{dx}{rx + x^3} = \frac{1}{2r}\left(-2\ln x + \ln(r + x^2)\right) =$$

$$\frac{1}{2r}\left(\ln\left(\frac{1}{x^2}\right)+\ln\left(r+x^2\right)\right)=\frac{1}{2r}\ln\left(\frac{r+x^2}{x^2}\right)\Rightarrow$$

$$2rt+C_2=\ln\left(\frac{r+x^2}{x^2}\right)\Rightarrow Ce^{2rt}=1+\frac{r}{x^2}\Rightarrow$$

$$\frac{Ce^{2rt}-1}{r}=\frac{1}{x^2}\Rightarrow x(t)=\pm\sqrt{\frac{r}{Ce^{2rt}-1}}$$

We know

$$x_0=\sqrt{\frac{r}{C-1}}\Rightarrow\frac{x_0^2}{r}=\frac{1}{C-1}\Rightarrow$$

$$\frac{r}{x_0^2}+1=C.$$

Since $x_0\neq 0$, this expression does not suck. Then,

$$x(t)=\pm\sqrt{\frac{r}{\left(\frac{r}{x_0^2}+1\right)e^{2rt}-1}}$$

This system has a finite-time singularity when

$$\left(\frac{r}{x_0^2}+1\right)e^{2rt} = 1 \Rightarrow e^{2rt} = \frac{1}{\frac{r}{x_0^2}+1} =$$

$$\frac{1}{\frac{1}{x_0^2}\left(r+x_0^2\right)} = \frac{x_0^2}{r+x_0^2} \Rightarrow 2rt = \ln\left(\frac{x_0^2}{r+x_0^2}\right)$$

$$\Rightarrow \boxed{t = \frac{1}{2r}\ln\left(\frac{x_0^2}{r+x_0^2}\right)} < \infty$$

2.5.4

We show that the IVP

$$\dot{x} = x^{1/3}, \quad x(0)=0$$

has infinitely many solutions.

This problem clearly has the trivial solution $x(t)=0$.

It has another solution $x(t) = \frac{2}{3}t^{3/2}$ by integration.

We next try to construct a solution that stays at $x=0$ until t_0, and then becomes nonzero.

Ansatz: $x(t) = \Theta(t-t_0)\cdot\frac{2}{3}(t-t_0)^{3/2}$

$$\dot{x} = \begin{cases} 0, & x < t_0 \\ t-t_0, & x > t_0, \end{cases}$$