We use linear stability analysis to classify the stability of the fixed points of the following systems.

2.4.1

$\dot{x} = x(1-x)$.

The fixed points of this system are $x^* = 0, 1$ by inspection.

We recall that for linear stability analysis, we are interested in the differential equation

$$\dot{\eta} = \eta \cdot f'(x^*)$$

and the sign of $f'(x^*)$ determines the stability of the fixed point.

$$f'(x) = \frac{d}{dx}(x - x^2) = 1 - 2x.$$

$f'(0) = 1 - 0 = 1 > 0$, so $x^* = 0$ is unstable.

$f'(1) = 1 - 2 = -1 < 0$, so $x^* = 1$ is stable.

2.4.2

$\dot{x} = x(1-x)(2-x) = (x-x^2)(2-x) = 2x - x^2 - 2x^2$
$+x^3 =$
$x^3 - 3x^2 + 2x$

By inspection, the fixed points are $x^* = 0, 1, 2.$

$f'(x) = 3x^2 - 6x + 2 \implies$

$f'(0) = 2 > 0$ unstable.

$f'(1) = 3 - 6 + 2 = -1 < 0$ stable.

$f'(2) = 3 \cdot 4 - 6 \cdot 2 + 2 = 2 > 0$ unstable.

2.4.3

$\dot{x} = \tan x.$

Fixed points are when $\tan x$ vanishes $\iff \sin x$ vanishes

$\implies x^* = z\pi,\ z \in \mathbb{Z}.$

$f'(x) = \sec^2 x = \dfrac{1}{\cos^2 x}$

$f'(x^*) = \dfrac{1}{\cos^2(z\pi)} = \dfrac{1}{(\pm 1)^2} = 1 > 0.$

All fixed points are unstable.

2.4.4

$\dot{x} = x^2(6-x)$

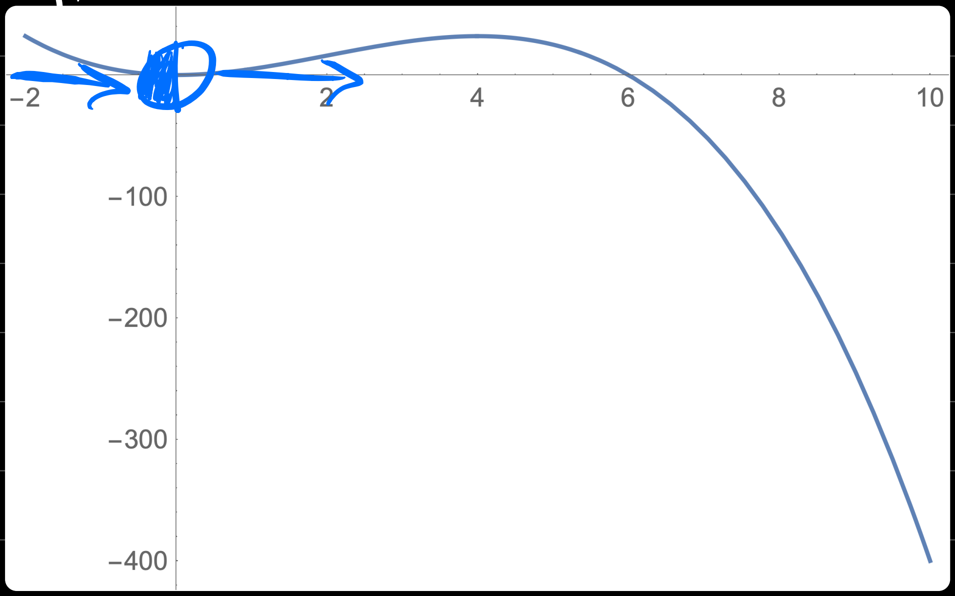
Fixed points: $x^* = 0, 6.$

$f(x) = 6x^2 - x^3 \Rightarrow f'(x) = 12x - 3x^2 \Rightarrow$

$f'(0) = 0$, inconclusive.

$f'(6) = 12 \cdot 6 - 3 \cdot 36 = -36 < 0$ stable.

Graphically, we have



Clearly, $x^* = 0$ is semi-stable.

2.4.5

$$\dot{x} = 1 - e^{-x^2}$$

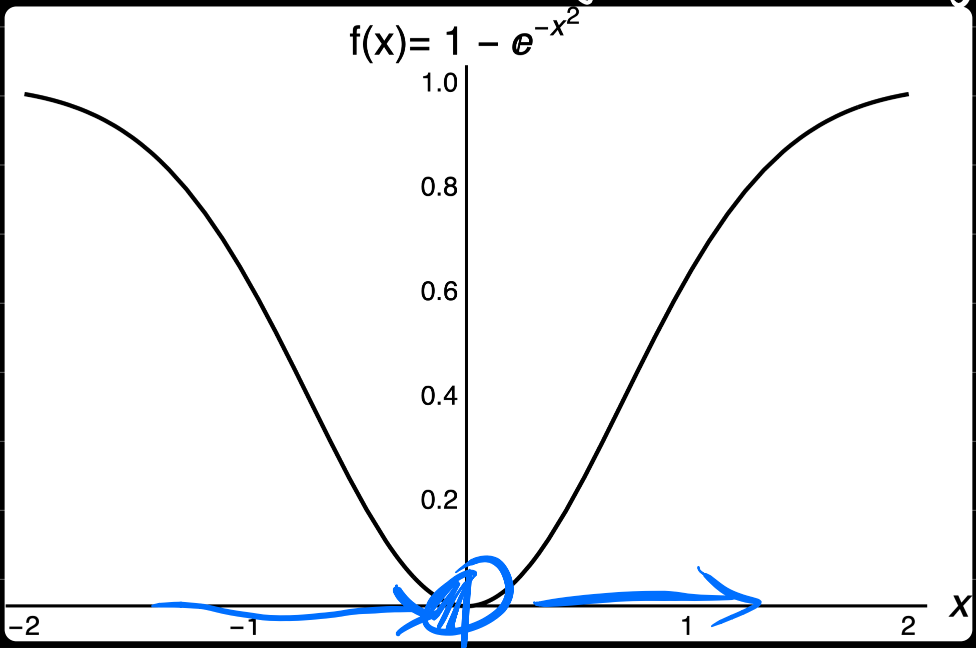
We find the fixed points of this system:

$$1 = e^{-x^2} \Rightarrow x^* = 0.$$

$$f'(x) = -e^{-x^2} \cdot -2x = 2xe^{-x^2}$$

$\Rightarrow f'(0) = 0$ inconclusive.

Hence, we plot $f(x)$ and analyze the fixed point graphically



Hence, again, $x^* = 0$ is semistable

2.4.9

We analyze the phenomenon of "critical slowing down" which is a result from statistical physics when a system settles down to equilibrium more slowly than usual during phase transitions.

a.) Let $\dot{x} = -x^3$. We find its analytical solution and show its long-term behavior.

$$\frac{dx}{dt} = -x^3 \Rightarrow \int \frac{dx}{-x^3} = t + C_1 = \frac{1}{2x^2}$$

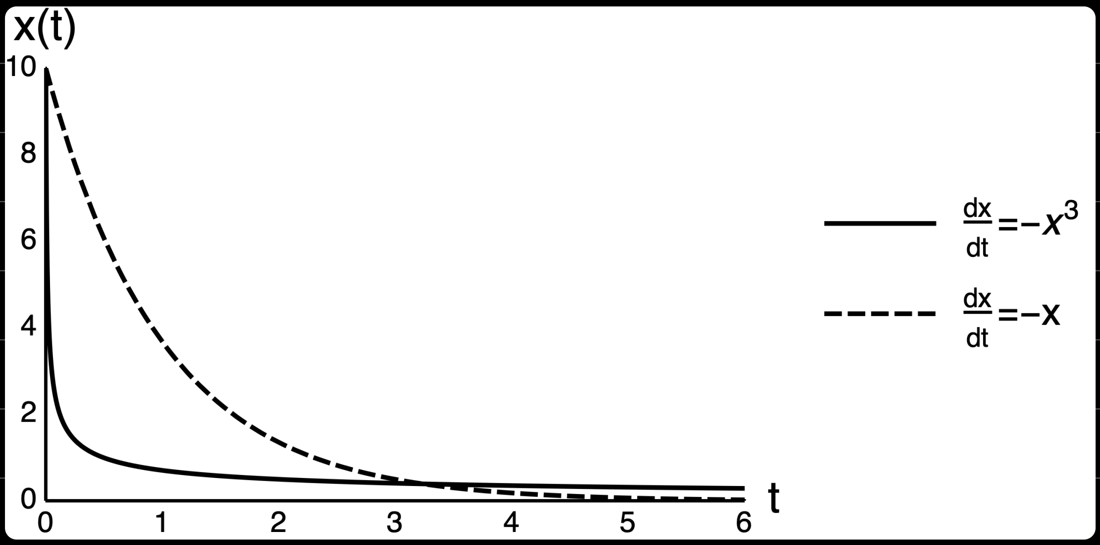
$$\Rightarrow 2x^2 = \frac{1}{t+C_1} \Rightarrow x(t) = \sqrt{\frac{1}{2t+C}}$$

Indeed, we confirm $\lim_{t\to\infty} x(t) = 0$, and the decay is relatively slow.

b.) Suppose $x_0 = 10$. Then

$$10 = \sqrt{\frac{1}{C}} \Rightarrow C = 1/100$$

We plot this, alongside the solution to $\dot{x} = -x$ (exponential decay):



We observe that the initial decay is quite fast, but it is soon passed by the exponential.