2.3.11

We consider the logistic equation

$$\dot{N} = rN(1-N/K)$$

a.) We solve this equation using separation

$$\frac{dN}{dt} = rN(1-N/K) \Rightarrow rt + C_1 = \int \frac{dN}{N(1-N/K)}$$

$$= \ln N - \ln(N-K) = \ln\left(\frac{N}{N-K}\right) \Rightarrow$$

$$\frac{N}{N-K} = C_2 e^{rt} \Rightarrow N = C_2 e^{rt} \cdot N - KC_2 e^{rt}$$

$$\Rightarrow N(1 - C_2 e^{rt}) = -KC_2 e^{rt} \Rightarrow$$

$$N(t) = \frac{KC_2 e^{rt}}{C_2 e^{rt} - 1} = \boxed{\frac{K}{1 - Ce^{-rt}} = N(t)}$$

b.) We now solve the logistic equation

$$\dot{N} = rN(1-N/K)$$

using the substitution $x = 1/N \Rightarrow \dot{x} = \frac{dx}{dN}\frac{dN}{dt}$

$$\Rightarrow \dot{N} = \frac{\dot{x}}{\frac{dx}{dN}} = \frac{\dot{x}}{\frac{-1}{N^2}} = \frac{-\dot{x}}{x^2} \Rightarrow$$

$$\frac{-\dot{x}}{x^2} = \frac{r}{x}\left(1 - \frac{1}{Kx}\right) \Rightarrow \dot{x} = rx\left(\frac{1}{Kx} - 1\right)$$

$$= r\left(\frac{1}{K} - x\right) = \frac{r}{K} - rx = \frac{dx}{dt} \Rightarrow$$

$$t + C_1 = \int \frac{dx}{\frac{r}{K} - rx} = -\frac{1}{r}\ln(1 - Kx) \Rightarrow$$

$$1 - Kx = C_2 e^{-rt} \Rightarrow x = \frac{1}{K} - Ce^{-rt} = \frac{1}{N}$$

$$\Rightarrow N = \frac{1}{1/K - Ce^{-rt}} = \boxed{\frac{K}{1 - De^{-rt}} = N(t)}$$

Initial value problem is trivial, so we omit it.

2.3.2

We consider a model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

This is an "auto catalytic" reaction, since the chemical X stimulates its own production. The back-reaction is also possible.

In chemistry, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. This is known as the Law of Mass Action.

We denote concentrations with lowercase letters, and assume $a = \text{const}$. Then, we have

$$\dot{x} = k_1 a x - k_{-1} x^2,$$

where $k_1, k_{-1} > 0$ are parameters known as rate constants.

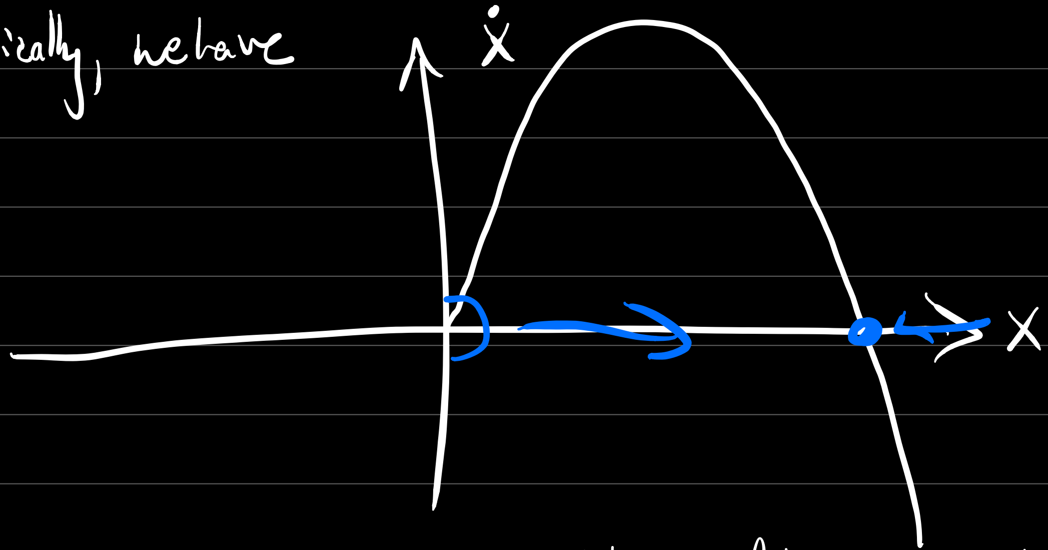
a.) We find and classify the fixed points of this system.

$$k_1 a x^* - k_{-1}(x^*)^2 = 0 = x^*(k_1 a - k_{-1} x^*)$$

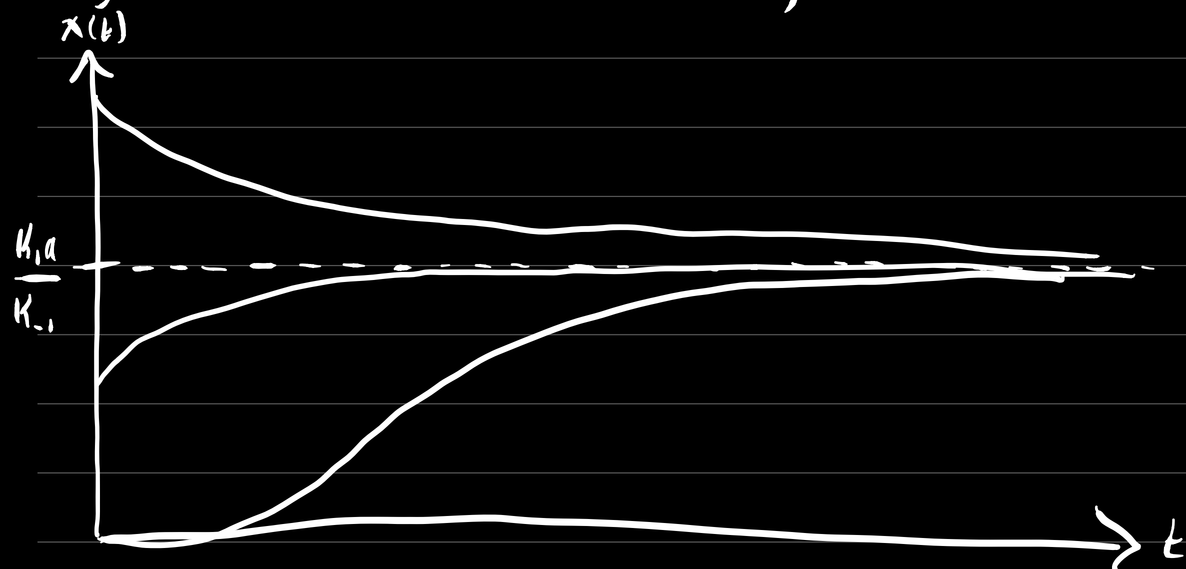
Hence, we have fixed points when

$\boxed{x^* = 0}$ and when $k_1 a - k_{-1} x^* = 0 \iff \boxed{x^* = \frac{k_1 a}{k_{-1}}}$

Graphically, we have



Thus, we see that $x^* = 0$ is unstable, and the other is stable.



2.3.5

$$\dot{X} = aX, \quad \dot{Y} = bY, \quad a > b > 0, \; X(0) = X_0,$$
$$Y(0) = Y_0.$$

a.) We expect $X(t)$ to dominate the population $X(t)+Y(t)$ over long times.

Let

$$x(t) = \frac{X(t)}{X(t)+Y(t)} \Rightarrow \dot{x} = \frac{1}{(X(t)+Y(t))^2}\left[\dot{X}(X+Y) - X(\dot{X}+\dot{Y})\right] = \frac{1}{(X+Y)^2}\left[\cancel{X\dot{X}} + \dot{X}Y - \cancel{X\dot{X}} - X\dot{Y}\right]$$

$$= \frac{X\cdot Y}{(X+Y)^2}(a-b).$$ Since all terms in this expression are positive, we conclude $\dot{x} > 0$ $\forall t$. That is, x increases monotonically in t.

We now wish to show $\lim_{t\to\infty} x(t) = 1$.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{X(t)}{X(t)+Y(t)} \overset{\text{l'Hôpital}}{=} \lim_{t\to\infty} \frac{\dot{X}}{\dot{X}+\dot{Y}} = \lim_{t\to\infty} \frac{aX}{aX+bY}$$