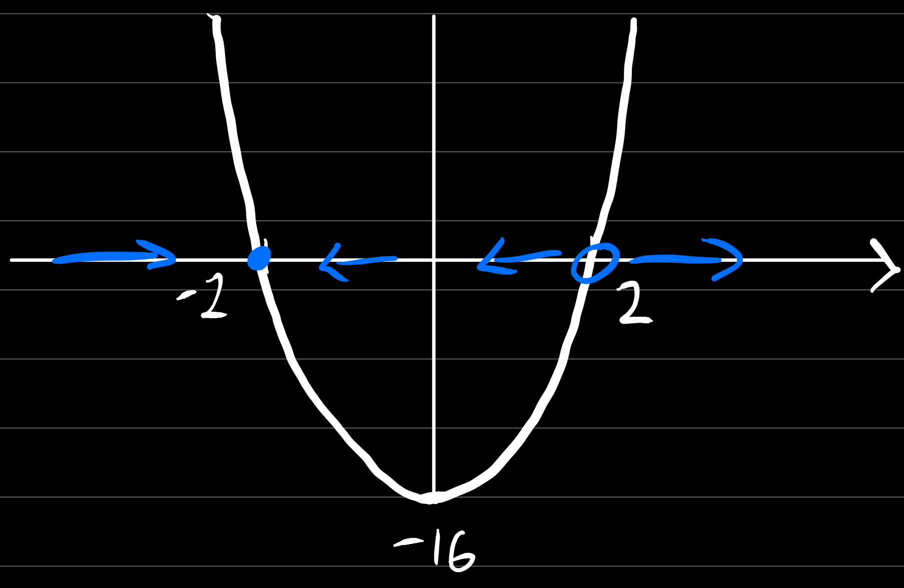
For the following 6 problems, we analyze the differential equations graphically.

- vector field
- fixed points
- stability
- $x(t)$ graphically
- $x(t)$ analytically?

2.2.1

Let $\dot{x} = 4x^2 - 16$.

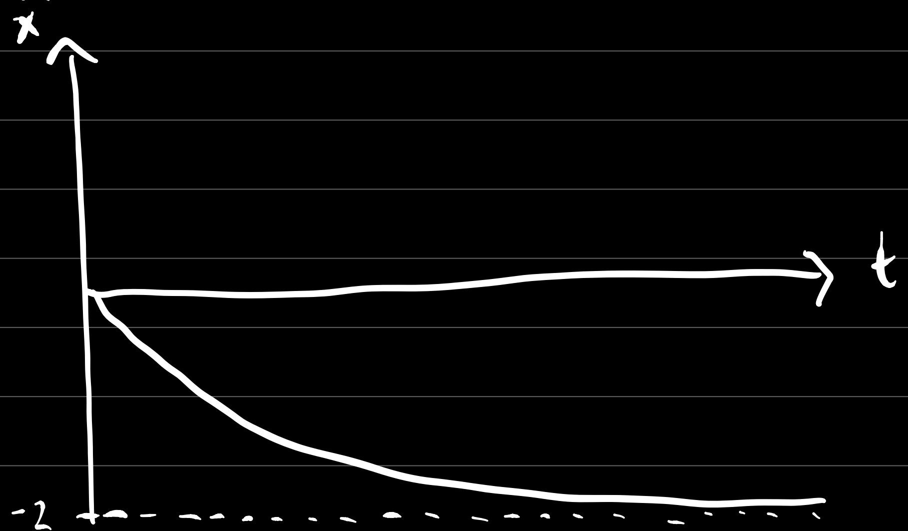
Then, we have an upward-opening parabola, shifted below the horizontal axis.



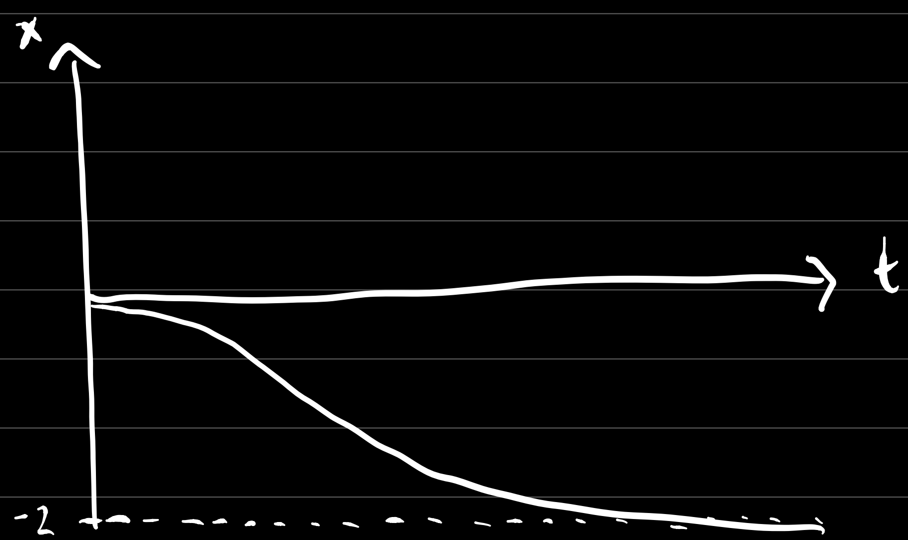
We have $x^* = \pm 2$, since x^* satisfies $f(x^*) = 0$.
From our vector field, we see that $x^* = 2$ is unstable, and $x^* = -2$

is stable.

Suppose $x_0 = 0$, then $x(t)$ behaves like



Suppose $x_0 = 1$, then



Lastly, we attempt to find $x(t)$ analytically.

$$\frac{dx}{dt} = 4x^2 - 16 \Rightarrow \frac{dx}{4x^2-16} = dt \Rightarrow$$

$$t = \int \frac{dx}{4x^2-16} = -\frac{1}{8}\tanh^{-1}\left(\frac{x}{2}\right) + C \Rightarrow$$

$$-8(t-C) = \tanh^{-1}(x/2) \Rightarrow$$

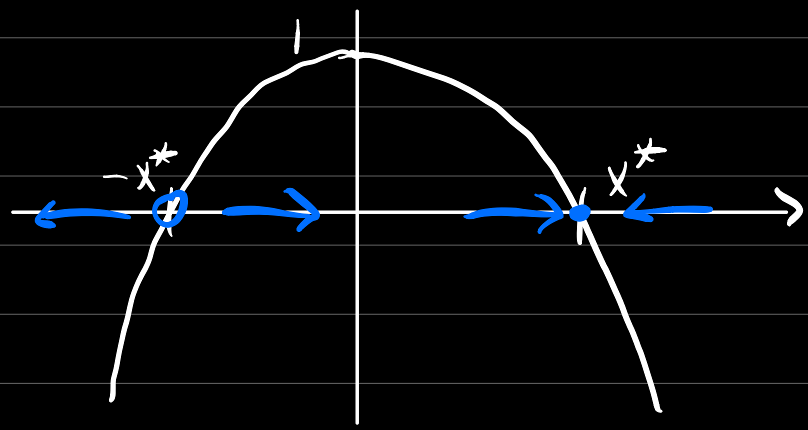
$$x(t) = 2\tanh\left(8(C-t)\right)$$

Something like that.

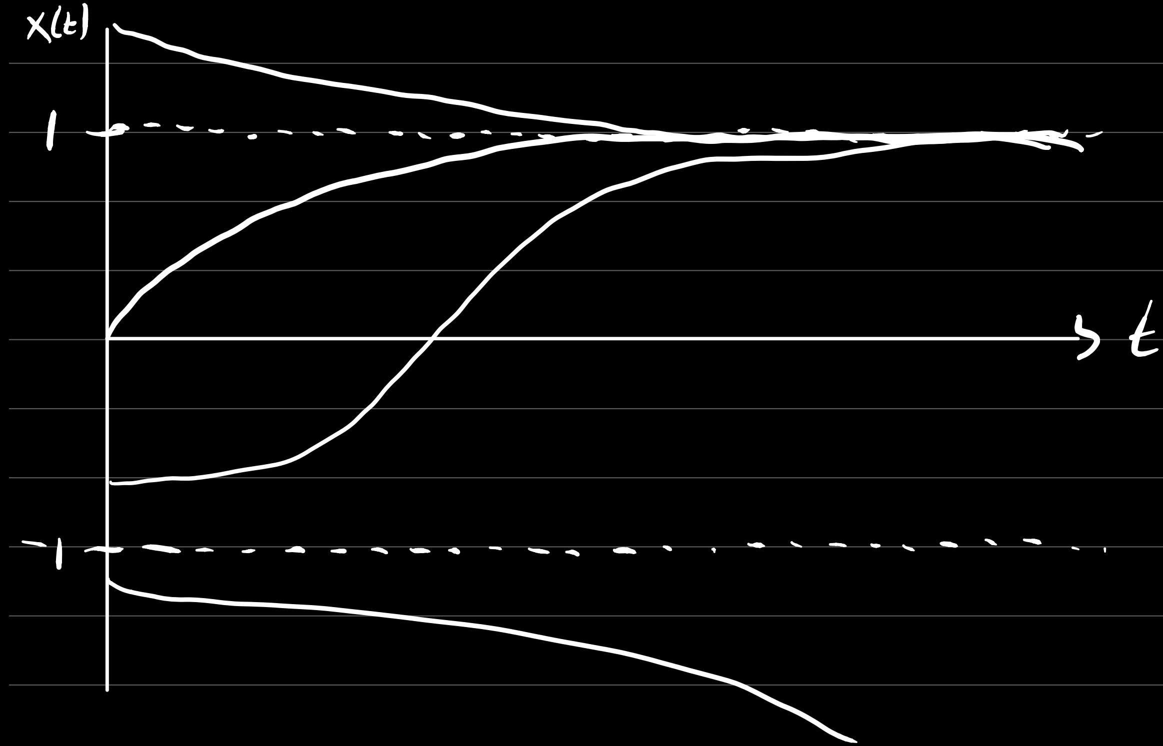
2.2.2

Let $\dot{x} = 1 - x^{14}$.

Given that x^{14} is even, we have



We observe that our fixed points are $x^* = \pm 1$; $x^* = -1$ is unstable, $x^* = 1$ is stable.



Lastly, we try to find $x(t)$ analytically: $\dot{x} = 1 - x^{14} \Rightarrow$

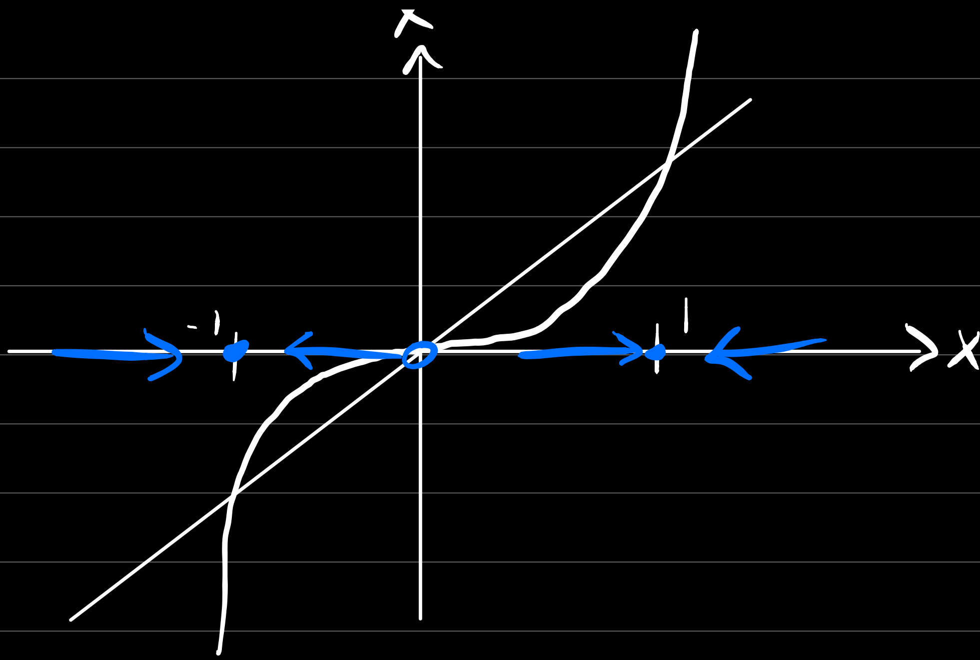
$$\int \frac{dx}{1 - x^{14}} = t$$

No easy solution.

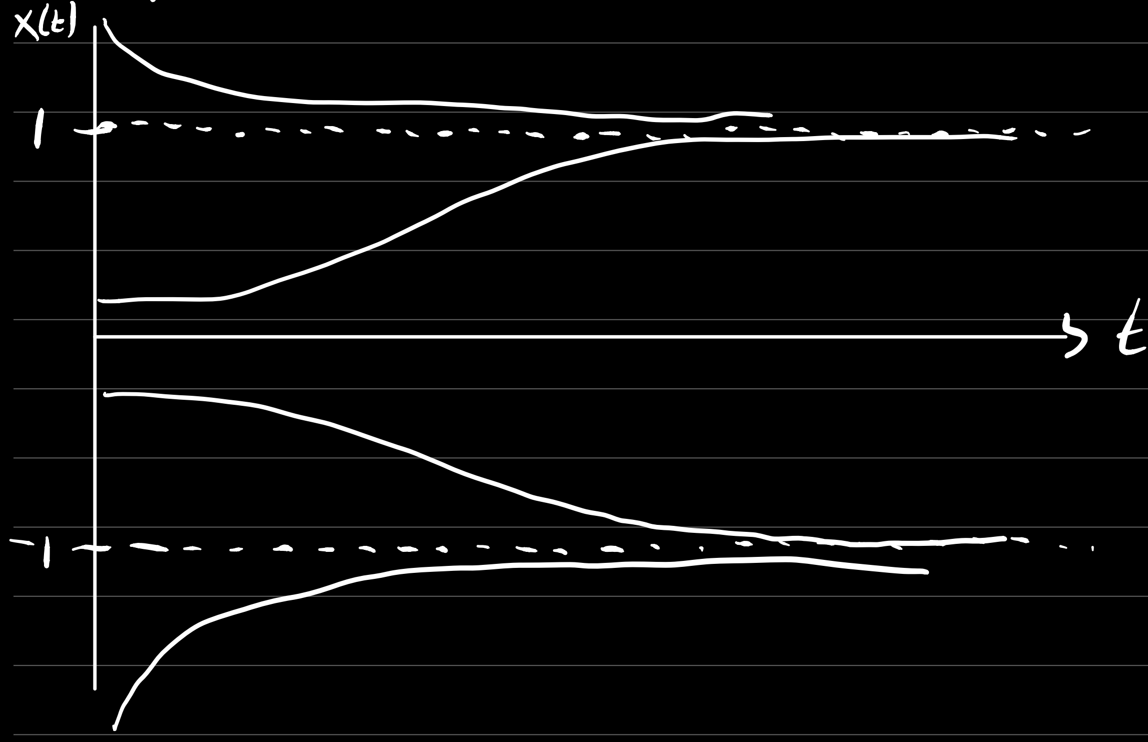
2.2.3

$\dot{x} = x - x^3$. We plot each term separately and compare their plots:

$\dot{x}$



Fixed points are $x^* = -1, 0, 1$: stable, unstable, stable.



To solve analytically, we must have

$$t = \int \frac{dx}{x - x^3} = \ln(x) - \frac{1}{2}\ln(1-x^2) + \tilde{C}$$

$$= \ln\left(\frac{x}{\sqrt{1-x^2}}\right) + \tilde{C} \Rightarrow e^{t-\tilde{C}} = \frac{x}{\sqrt{1-x^2}} = Ce^{t}$$

$$\Rightarrow \sqrt{1-x^2}\,Ce^{t} = x \Rightarrow x^2 = (1-x^2)C^2e^{2t} = De^{2t} - Dx^2e^{2t}$$

$$\Rightarrow x^2\left(1+De^{2t}\right) = De^{2t} \Rightarrow x^2 = \frac{De^{2t}}{1+De^{2t}}$$

$$= \frac{1}{1+De^{-2t}} \Rightarrow x = \pm\frac{1}{\sqrt{1+De^{-2t}}}.$$

$$x(0) = x_0 = \pm\frac{1}{\sqrt{1+D}} \Rightarrow x_0^2 = \frac{1}{1+D}$$

$$\Rightarrow \frac{1}{x_0^2} = 1+D \Rightarrow D = \frac{1}{x_0^2} - 1$$

$$\Rightarrow x(t) = \pm\frac{1}{\sqrt{1+\left(\frac{1}{x_0^2}-1\right)e^{-2t}}}.$$

Strogatz has

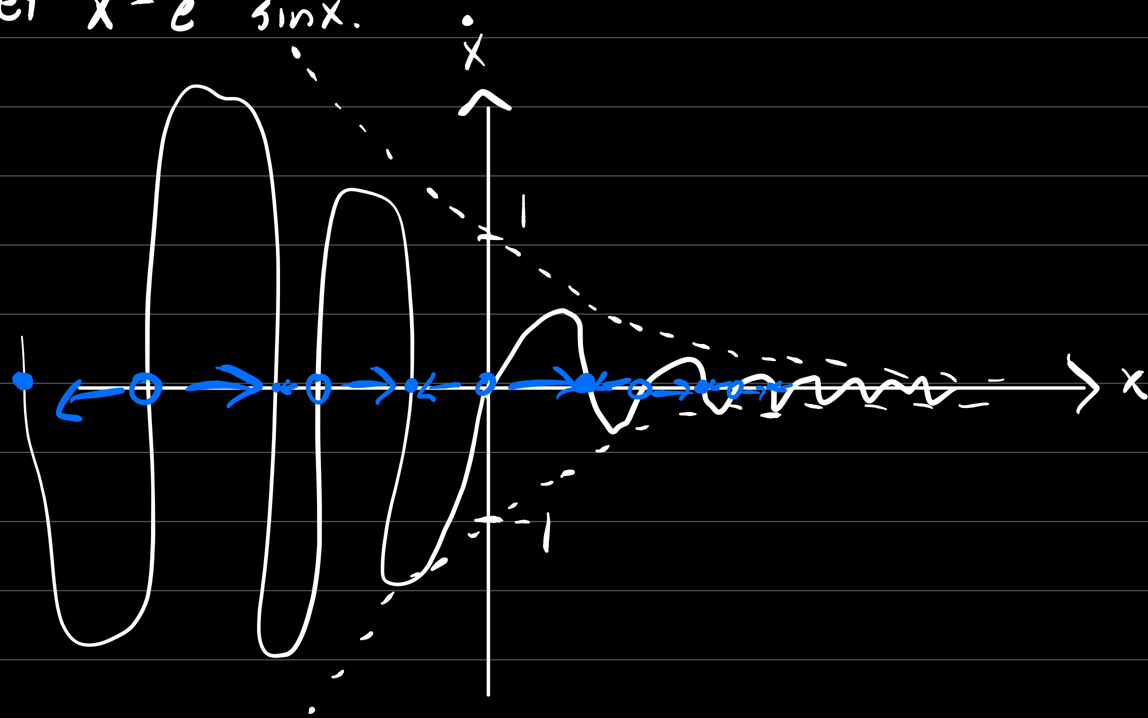
$$x(t) = \frac{\pm e^{t}}{\sqrt{\frac{1}{x_0^2} + e^{2t} - 1}} = \frac{\pm e^{t}}{\sqrt{e^{2t}\left(\frac{1}{x_0^2}e^{-2t} + 1 - e^{-2t}\right)}} =$$

$$\pm \frac{\cancel{e^{t}}}{\cancel{e^{t}}\sqrt{1 + e^{-2t}\left(\frac{1}{x_0^2} - 1\right)}}$$

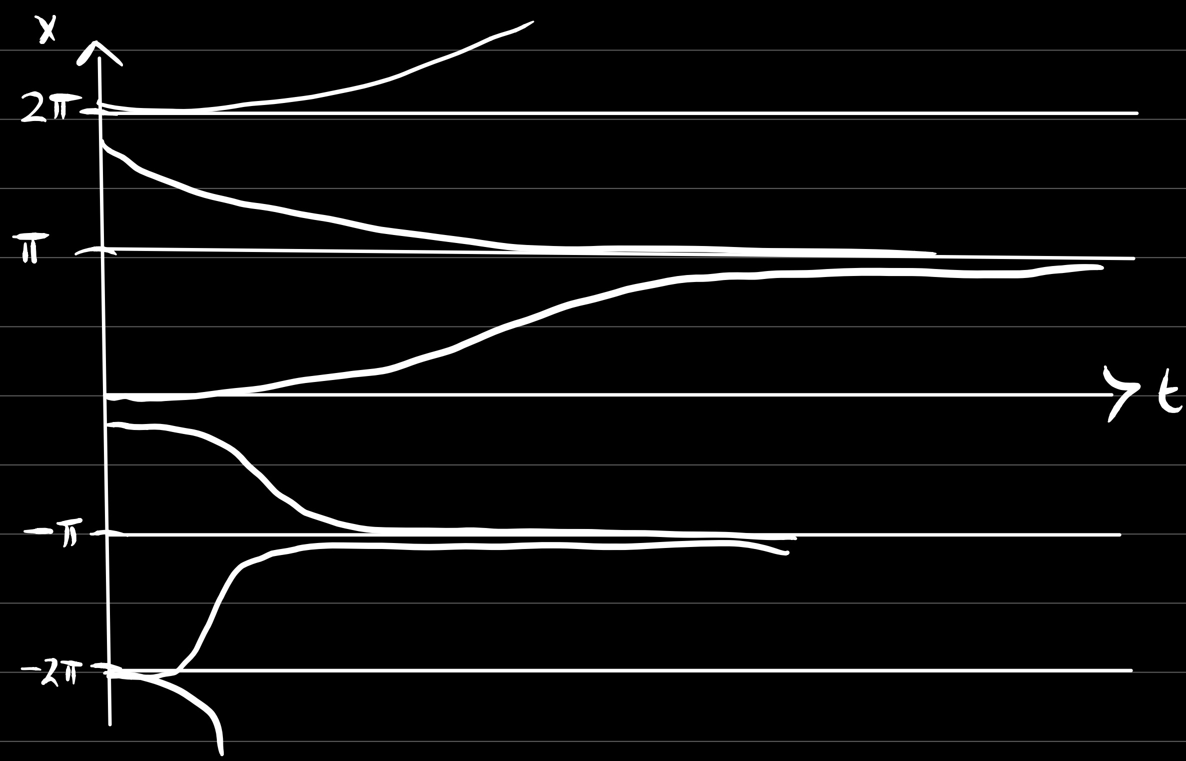
Check.

2.2.4

Let $\dot{x} = e^{-x}\sin x$.



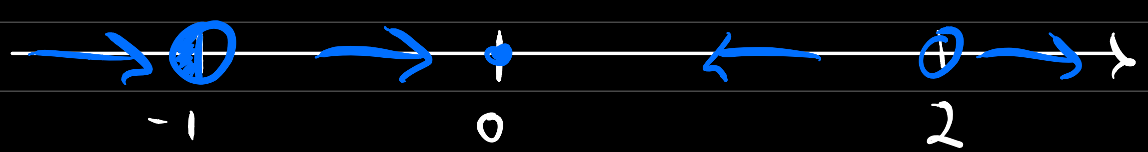
$\dot{x}$ has fixed points $x^* = z\pi$, $z \in \mathbb{Z}$. The stable points are $x^* = \pi + 2\pi z$, $z \in \mathbb{Z}$. The unstable points are $x^* = 2\pi z$, $z \in \mathbb{Z}$.



No analytical solution.

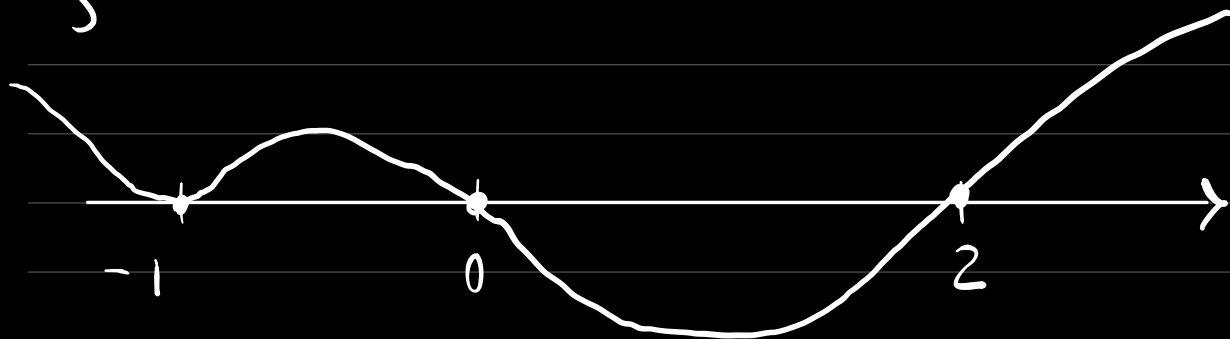
2.2.8

Given the following phase portrait, we find an associated system



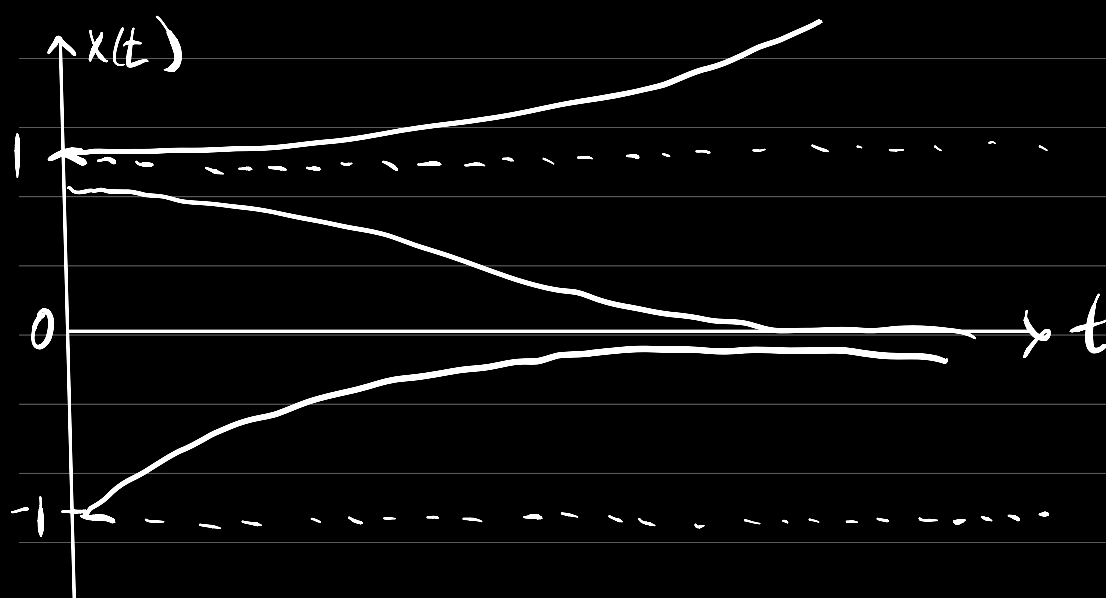
We recall that the circles here represent fixed points x^* of the flow, which satisfy $f(x^*) = 0$.

Hence, $f(x)$ has 3 zeros, and its sign is determined by the direction of flow:



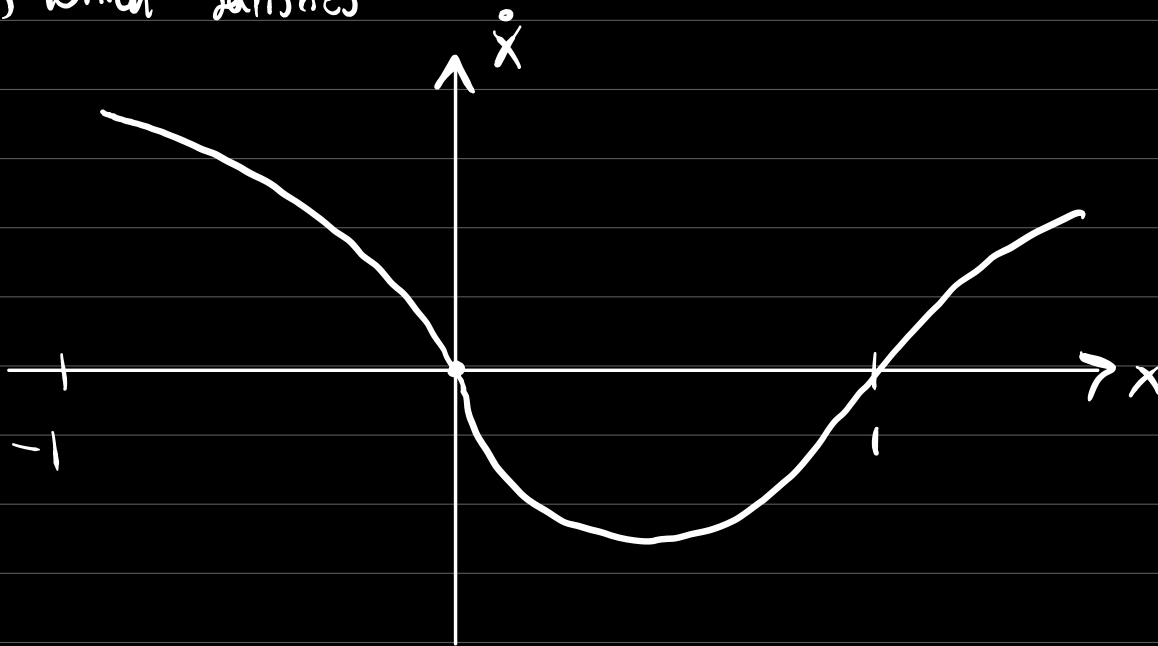
2.2.9

We find a system $\dot{x} = f(x)$ that has the following qualitative solutions



We have 3 fixed points. $x^* = 0$ is stable. $x^* = 1$ is unstable. We only know part of $x^* = -1$ behavior.

Given the apparent acceleration from the center solution, we infer nonmonotonicity between 0 and 1. Hence, we propose $f(x)$ which satisfies



2.2.10

For the following criteria, we find a system $\dot{x} = f(x)$, or else we explain why such a system cannot exist. We assume $f(x)$ is smooth.

a.) Every real number is a fixed point. That is, $\forall x\ f(x) = 0$. Thus, we have

$$\boxed{\dot{x} = 0}$$

b.) The only fixed points are the integers. That is,

$$\forall x \in \mathbb{Z} \ f(x) = 0; \ \forall y \notin \mathbb{Z} \ f(y) \neq 0.$$

We propose the following function

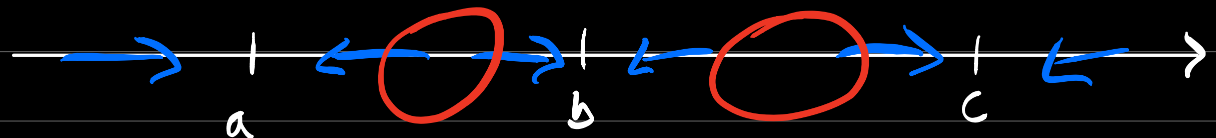
$$\boxed{f(x) = \sin(\pi x)}$$

We observe that $\forall z \in \mathbb{Z} \ f(z) = \sin(\pi z) = 0$
$\forall w \notin \mathbb{Z} \ f(w) = \sin(\pi w) \neq 0$

c.) There are precisely 3 fixed points and all of them are stable.

This is impossible. We demonstrate this with a phase portrait.
Suppose our 3 fixed points are a, b, c satisfying $a < b < c$.

Our phase portrait is thus



Without 2 additional fixed points, this condition is not possible.

d.) There are no fixed points. That is, $\forall x \in \mathbb{R}\ f(x) \neq 0$. This is easily satisfied by

$$\boxed{f(x) = \text{const.} > 0}$$

(among many other choices.)

e.) There are precisely 100 fixed points. Let $\{x_k^*\}_{k=1}^{100}$ be 100 distinct real numbers. Then the following function has precisely 100 fixed points:

$$\boxed{f(x) = \prod_{k=1}^{100} (x - x_k^*)}$$

2.2.11

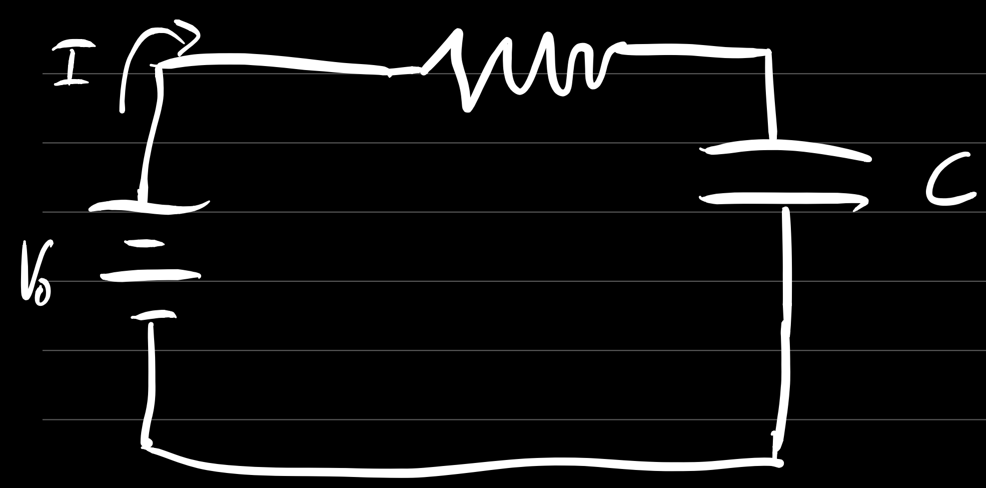
We obtain the analytical solution for the charge $Q(t)$ on a capacitor in an RC circuit.

We recall that capacitors satisfy

$$Q = CV$$

We next draw the circuit:

R



We know that the voltage drop around this circuit is 0. Hence, we have

$$V_0 - IR - \frac{Q(t)}{C} = 0$$

Since $I = \dot{Q}$, we have

$$V_0 - \dot{Q}R - Q/C = 0 \Rightarrow$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}.$$

We also assert that a switch is closed at $t=0$, and hence

$$Q(0) = 0$$

This equation is separable. Let $V_0/R \equiv \alpha$, $\frac{-1}{RC} \equiv \beta$.

Then,

$$\frac{dQ}{dt} = \alpha + \beta Q \Rightarrow \int \frac{dQ}{\alpha+\beta Q} = \int dt = t.$$

Let $u = \alpha + \beta Q$, $du = \beta dQ$ $\Rightarrow t = \int \frac{du/\beta}{u} = \frac{1}{\beta} \ln u + C \Rightarrow$

$$u = e^{\beta(t-C)} = Ae^{\beta t} = \alpha + \beta Q \Rightarrow$$

$$Q(t) = \frac{1}{\beta}\left[Ae^{\beta t} - \alpha\right] = -RCAe^{-t/RC} + \frac{V_0}{R}RC$$

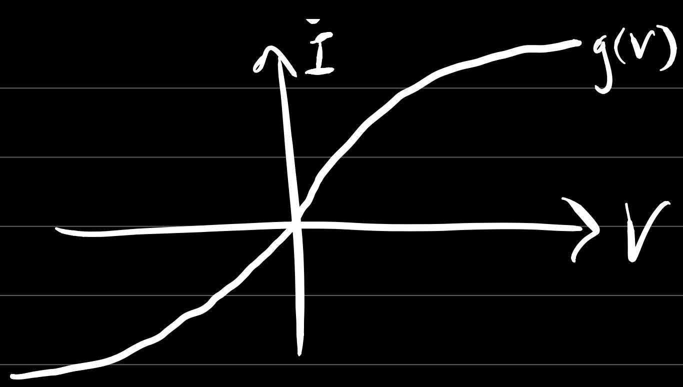
$= V_0 C - RCAe^{-t/RC}$. To solve for A, we impose our initial condition

$$Q(0) = 0 = V_0 C - RCA \Rightarrow V_0 \cancel{C} = R\cancel{C}A \Rightarrow$$

$$A = \frac{V_0}{R} \Rightarrow \boxed{Q(t) = V_0 C(1 - e^{-t/RC})}$$

2.2.12

We now consider an RC circuit with a nonlinear resistor. The resistor allows current flow $I_R = g(V)$ for some function g with shape



Given that the voltage drop around the circuit is 0, we have

$$V_0 - V_R - Q/C = 0.$$

We observe that g is invertible, and hence $V_R = g^{-1}(I)$

$$\Rightarrow V_0 - g^{-1}(\dot{Q}) - Q/C = 0 \Rightarrow$$

$$g^{-1}(\dot{Q}) = V_0 - Q/C \Rightarrow \boxed{\dot{Q} = g(V_0 - Q/C)}$$

Given that we have found the circuit equation, we find its fixed points.

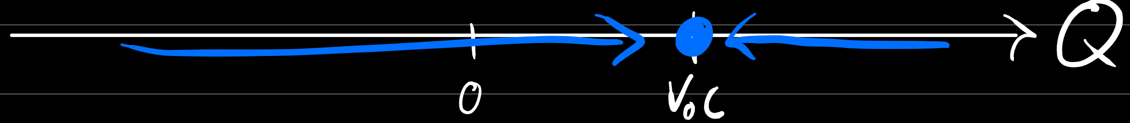
$$g(x) = 0 \Leftrightarrow x = 0$$

Hence, the only fixed point occurs when $V_0 - Q^*/C = 0$

$$\Leftrightarrow \boxed{Q^* = V_0 C}$$ just like in a standard RC circuit.

Moreover, $V_0 < Q^*/C \Rightarrow g < 0$, $V_0 > Q^*/C \Rightarrow g > 0$

Thus, our phase portrait is



Hence, we conclude that $Q^* = V_0 C$ is a stable, global fixed point, just like in the case of the linear resistor.

Hence, qualitatively, the nonlinear resistor doesn't change anything about the circuit.

2.2.13

We consider the differential equation for a falling body under drag:

$$m\dot{v} = mg - kv^2$$

a.) We find the analytical solution to this system, with the IC

$$v(0) = 0$$

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 = g - hv^2 \implies$$

$$t = \int \frac{dv}{g - hv^2} = \frac{1}{\sqrt{gh}} \tanh^{-1}\left(\sqrt{\frac{h}{g}}\, v\right) = \sqrt{\frac{m}{gk}} \tanh^{-1}\left(\frac{k}{\sqrt{mg}} v\right)$$

$$\Rightarrow \boxed{v(t) = \sqrt{\frac{mg}{K}} \tanh\left(\sqrt{\frac{gK}{m}}\, t\right)}$$, which already satisfies $v(0) = 0$ (hence the integration constant that we omitted was 0).

Given that $\lim_{x\to\infty} \tanh x = 1$, $\lim_{t\to\infty} \sqrt{\frac{mg}{K}} \tanh\left(\sqrt{\frac{gK}{m}}\, t\right) =$

$$\boxed{\sqrt{\frac{mg}{K}} = v_{terminal}}$$

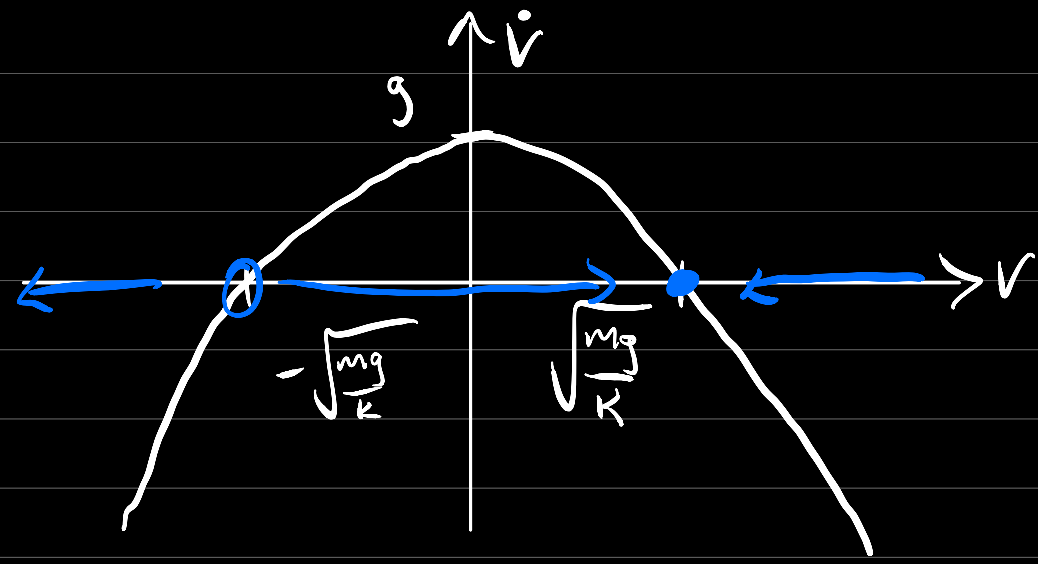
c.) We now analyze this same system graphically:

$$\dot{v} = g - \frac{K}{m} v^2$$

We have fixed points when

$$g = \frac{K}{m} {v^*}^2 \Rightarrow v^* = \pm\sqrt{\frac{mg}{K}}$$

Hence, graphically, we have



Clearly, our solution is nonphysical for $v < -\sqrt{\frac{mg}{k}}$. But we got a stable fixed point at the body's (downward) terminal velocity, and our results agree with our analytical solution.