Suppose $\dot{x} = \sin x$

2.1.1

We find all fixed points of this flow. A fixed point of a flow is a point satisfying

$$\dot{x} = 0$$

Here, we have $\dot{x} = \sin x = 0 \Rightarrow x = \pm n\pi, n \in \mathbb{N}$.

That is, the fixed points of this flow occur at all multiples of π on the real line.

2.1.2

At which points x does the flow have the greatest velocity to the right?

We wish to maximize $\dot{x}(x)$. Equivalently, we wish to find points such that

$$\frac{d\dot{x}}{dx} = 0, \quad \frac{d^2\dot{x}}{dx^2} < 0$$

$$\frac{d\dot{x}}{dx} = \frac{d}{dx}\sin x = \cos x = 0 \Rightarrow x = \pm\frac{(2n+1)\pi}{2}, n \in \mathbb{N}.$$

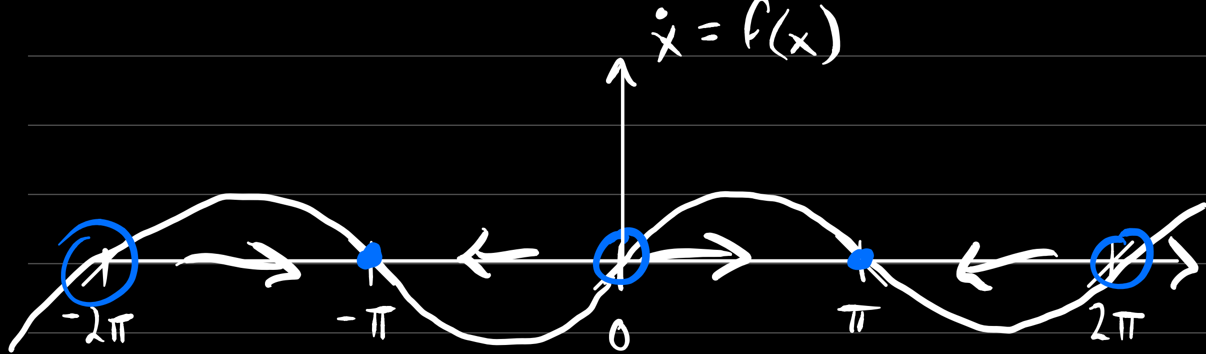
$$\frac{d^2\dot{x}}{dx^2} = \frac{d}{dx}\cos x = -\sin x.$$

Thus, we require $-\sin\left(\pm\frac{(2n+1)\pi}{2}\right) < 0 \implies$

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots, \frac{-3\pi}{2}, \frac{-7\pi}{2}, \ldots$$

That is, $\boxed{x = \frac{\pi}{2} \pm 2n\pi,\ n \in \mathbb{N}.}$

Indeed, we could have skipped this analysis entirely and simply looked at the plot of the flow



Clearly the flow moves to the right when $\dot{x} > 0$, and does so most quickly at the max of $\dot{x}$. For $\sin(x)$, this occurs at $\pi/2$ and all $2\pi n$ translations of $\pi/2$.

2.1.3|

We now find the flow's acceleration $\ddot{x}(x)$.

a.)

Given $\dot{x} = f(x)$, $\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt} f(x) =$

$$\frac{d}{dt} f(x(t)) = \frac{df}{dx}\frac{dx}{dt}.$$

So in our case,

$$\ddot{x} = \frac{d}{dx}\sin x \cdot \dot{x} = \cos x\,\dot{x} = \boxed{\cos x \sin x = \ddot{x}}$$

b.) Where does the flow have maximum positive acceleration?

This occurs where

$$\frac{d\ddot{x}}{dx} = 0, \quad \frac{d^2\ddot{x}}{dx^2} < 0$$

$$\frac{d\ddot{x}}{dx} = \frac{d}{dx}\sin x\cos x = \cos^2 x - \sin^2 x = 0$$

$$\Longleftrightarrow \cos^2 x = \sin^2 x \Longrightarrow \cos x = \pm \sin x$$

$$\Longrightarrow x = \pm\frac{(2n+1)\pi}{4}, \ n \in \mathbb{N}.$$

Next, we evaluate the concavity of the acceleration

$$\frac{d^2\ddot{x}}{dx^2} = \frac{d}{dx}\left(\cos^2 x - \sin^2 x\right) = -2\cos x\sin x - 2\sin x\cos x$$

$$= -4\sin x\cos x < 0 \Rightarrow -4\sin\left(\frac{\pm(2n+1)\pi}{4}\right)\cos\left(\frac{\pm(2n+1)\pi}{4}\right)$$

< 0, which means $\sin\left(\frac{\pm(2n+1)\pi}{4}\right)\cos\left(\frac{\pm(2n+1)\pi}{4}\right) > 0.$

This happens when both functions have the same argument. That is, the points of maximum positive acceleration are

$$\boxed{x = \frac{\pi}{4} + z\pi,\ z \in \mathbb{Z}}$$

2.1.4

We can in fact solve $\dot{x} = \sin x$ exactly. We get

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

a.) We solve for $x(t)$ given $x_0 = \pi/4$, and solve for its limiting behavior.

We have

$$t = \ln\left|\frac{\csc(\pi/4) + \cot(\pi/4)}{\csc(x) + \cot(x)}\right| = \ln\left|\frac{1+\sqrt{2}}{\csc(x)+\cot(x)}\right|.$$

Using the trig identity $\frac{1}{\csc(x)+\cot(x)} = \frac{\sin x}{1+\cos x}$, we have $t = \ln\left|(1+\sqrt{2})\cdot\frac{\sin x}{1+\cos x}\right| = \ln(1+\sqrt{2}) + \ln\left|\frac{\sin x}{1+\cos x}\right| \Rightarrow$

$$t - \ln(1+\sqrt{2}) = \ln\left|\frac{\sin x}{1+\cos x}\right| \Rightarrow e^{t-\ln(1+\sqrt{2})} = \left|\frac{\sin x}{1+\cos x}\right|$$

$= \left|\tan\left(\frac{x}{2}\right)\right|$ by another trig identity. So, we have

$$\frac{e^t}{1+\sqrt{2}} = |\tan(x/2)|.$$

To agree with Strogatz, we ignore the absolute value, and derive

$$\boxed{x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)}$$

In the limit, $\lim_{t\to\infty} x(t) = 2\cdot\tan^{-1}(\infty) = \pi$.

b.) We now perform this analysis for arbitrary x_0. We start with

$$t = \ln\left(\frac{\csc x_0 + \cot x_0}{\csc(x) + \cot(x)}\right) = \ln(\csc x_0 + \cot x_0) + \ln\left(\tan\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow \boxed{2\tan^{-1}\left(\frac{e^{t}}{\csc x_0 + \cot x_0}\right) = x(t)}$$

Is our general solution.

2.1.5

We find a mechanical system approximately governed by $\dot{x} = \sin x$.

a.) We know the pendulum in a gravitational field is governed by

$$\ddot{x} = -\alpha \sin x$$

where x is the angle from the vertical.

b.) $x^* = 0$ is a stable fixed point because, if perturbed, it will settle back to $x = 0$. But $x^* = \pi$ is unstable because, if perturbed, it will come to rest at $x = 0$.