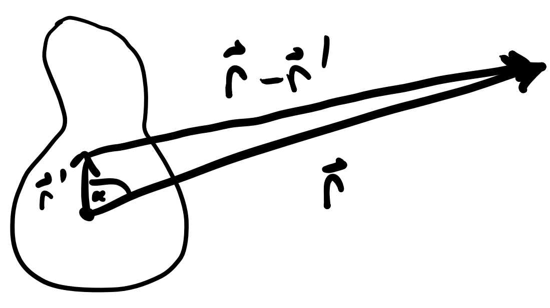
Exercise: We derive the multipole expansion in electrostatics.

(Source: Griffiths Electrodynamics, Section 3.4)

The electrostatic potential due to a charge distribution $\rho(\vec{r})$ is given by

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

where $\vec{r}$ is the field coordinate and $\vec{r}\,'$ is the source coordinate.



We derive a series expansion for $1/|\vec{r}-\vec{r}\,'|$.

$$|\vec{r}-\vec{r}\,'|^2 = (\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,') = r^2 - 2rr'\cos\alpha + r'^2 =$$

$$r^2\left(1-2\left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\right) = r^2\left[1+\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\alpha\right)\right] \equiv r^2(1+\varepsilon).$$

If we consider a field point distant from a compact charge distribution,

$$\left(\frac{r'}{r}\right) \ll 1 \Rightarrow \varepsilon \ll 1,$$

so we may Taylor expand in ε

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}(1+\varepsilon)^{-1/2} \approx \frac{1}{r}\left[1 - \frac{1}{2}\varepsilon + \frac{3}{4}\frac{\varepsilon^2}{2!} - \frac{15}{8}\frac{\varepsilon^3}{3!} + \cdots\right]$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 -\right.$$

$$\left.\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right] =$$

$$\frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \cdots\right]$$

$$= \frac{1}{r}\left[P_0(\cos\alpha) + \left(\frac{r'}{r}\right)P_1(\cos\alpha) + \left(\frac{r'}{r}\right)^2 P_2(\cos\alpha) + \left(\frac{r'}{r}\right)^3 P_3(\cos\alpha) + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha),$$

where $P_n(\cos\alpha)$ is the n^{th} Legendre polynomial.

Substituting into our initial integral, we derive the multipole expansion:

$$\boxed{\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\alpha)\,\rho(\vec{r}\,')\,d^3r'}$$

Writing out the first few terms, we have

$$\Phi(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\vec{r}\,')\,d^3r' + \frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\alpha\,\rho(\vec{r}\,')\,d^3r'$$

$$= \frac{Q}{4\pi\varepsilon_0 r} + \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\underbrace{\int \vec{r}\,'\rho(\vec{r}\,')\,d^3r'}_{\text{"dipole moment"}}$$

"monopole moment"

(i.e., the charge)