Exercise: We show that 1D symmetric probability distributions have the equal mean and median.

(Source: undergraduate stat mech)

Consider a 1D symmetric probability distribution $\rho(x) = \rho(-x)$. This distribution has mean 0, since the mean is

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$

and $x\rho(x)$ is an odd function under $x \mapsto -x$. The median x^* of a distribution is defined by

$$\frac{1}{2} = \int_{-\infty}^{x^*} \rho(x)dx \, .$$

For a symmetric distribution,

$$\int_{-\infty}^{0} \rho(x)dx = \int_{0}^{\infty} \rho(x)dx$$

$\hookrightarrow$ Proof

$$\int_{-\infty}^{0} \rho(x)dx = -\int_{0}^{-\infty} \rho(x)dx = \int_{0}^{\infty} \rho(x)d(-x) = \int_{0}^{\infty} \rho(-x)dx$$

$$= \int_{0}^{\infty} \rho(x)dx \qquad \blacksquare$$

So, it follows that

$$1 = \int_{-\infty}^{\infty} \rho(x)dx = \int_{-\infty}^{0} \rho(x)dx + \int_{0}^{\infty} \rho(x)dx = 2\int_{-\infty}^{0} \rho(x)dx \Rightarrow$$

$$\int_{-\infty}^{0} \rho(x)dx = \frac{1}{2} \Rightarrow x^* = 0.$$

Thus, we have shown that symmetric 1D distributions have mean zero and median zero, and therefore the mean equals the median.

$\blacksquare$