Exercise: We derive the mean and variance of the binomial distribution.

(Source: undergrad stat mech)

We recall that the binomial distribution

$$p(k) = N!/k!(N-k)! \; p^k (1-p)^{N-k}$$

gives the probability of observing k "successes" when performing N independent binary experiments, each with success probability p.

Let us call $q \equiv 1-p$. Then the binomial theorem says that

$$(p+q)^N = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} p^k q^{N-k} \equiv F(p,q),$$

where we have defined the "moment generating function" F. Clearly,

$F(p,q) = 1$. This shows that the binomial distribution is normalized.

Next, we compute

$$\frac{\partial F}{\partial p} = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} k p^{k-1} q^{N-k} = N(p+q)^{N-1} = N.$$

It follows that

$$p\frac{\partial F}{\partial p} = \boxed{\langle K\rangle = Np}$$

This is the mean of the binomial distribution.

We now compute the variance of the binomial distribution.

$$\sigma^2 = \langle (K-\langle K\rangle)^2\rangle = \langle K^2\rangle - \langle K\rangle^2$$

$$\frac{\partial^2 F}{\partial p^2} = \sum_{K=0}^{N} \frac{N!}{K!(N-K)!}(K^2-K)p^{K-2}q^{K} = N(N-1)(p+q)^{N-2}$$

$$= N^2 - N \Rightarrow p^2\frac{\partial^2 F}{\partial p^2} = \langle K^2\rangle - \langle K\rangle$$

$$= p^2N^2 - p^2N = \langle K\rangle^2 - p^2 N \Rightarrow$$

$$\langle K^2\rangle - \langle K\rangle^2 = Np - Np^2 = Np(1-p) \Rightarrow$$

$$\boxed{\sigma^2 = Np(1-p)}$$