<u>Exercise</u> | We derive the Larmor precession of an electron's magnetic moment in the presence of an external magnetic field.

(Source: Griffiths QM, example 4.3, p. 322.)

The electron's magnetic moment $\vec{\mu}$ is proportional to its spin

$$\vec{\mu} = \gamma \vec{S}.$$

We consider an electron in an external magnetic field $\vec{B}$, which we choose to be oriented in the z-direction:

$$\vec{B} = B\hat{z}.$$

The electron will experience a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ which tries to align the magnetic moment with the field, so, classically, the energy of the configuration is given by

$$U = \int \tau \, d\theta = \int_{\pi/2}^{\theta} \mu B \sin\tilde{\theta} \, d\tilde{\theta} = -\mu B \cos\theta = -\vec{\mu} \cdot \vec{B}$$

It follows that quantum mechanically, the electron will be described by the Hamiltonian

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\gamma B \hat{S}_z$$

We now solve the Schrödinger equation.

First, we solve the time-independent Schrödinger equation. Clearly, the eigenstates of this Hamiltonian are simply the eigenstates of $\hat{S}_z$, $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, with corresponding eigenvalues $\mp \gamma B\hbar/2$.

The solution to the time-dependent Schrödinger equation will be given by

$$|\Psi(t)\rangle = a e^{-iE_\uparrow t/\hbar} |\uparrow\rangle + b e^{-iE_\downarrow t/\hbar} |\downarrow\rangle,$$

where $\begin{pmatrix}a\\b\end{pmatrix} = |\Psi(0)\rangle$. Normalization requires

$|a|^2 + |b|^2 = 1$, so we choose $a \equiv \cos\alpha/2$, $b \equiv \sin\alpha/2$, which implies that

$$|\Psi(t)\rangle = \cos\alpha/2 \, e^{i\gamma Bt/2} |\uparrow\rangle + \sin\alpha/2 \, e^{-i\gamma Bt/2} |\downarrow\rangle.$$

We now show that this state exhibits precission of the electron's spin (and hence magnetic moment) about the magnetic field by computing $\langle \vec{S} \rangle$.

$$\langle S_x \rangle = \left(\cos\alpha/2 \, e^{-i\gamma Bt/2} \langle\uparrow| + \sin\alpha/2 \, e^{i\gamma Bt/2} \langle\downarrow|\right) \cdot$$

$$S_x \left(\cos\alpha/2 \, e^{i\gamma Bt/2} |\uparrow\rangle + \sin\alpha/2 \, e^{-i\gamma Bt/2} |\downarrow\rangle\right) =$$

$$\cos^2\alpha/2\,\langle\uparrow|S_x|\uparrow\rangle + \cos\alpha/2\,\sin\alpha/2\,e^{-i\gamma Bt}\langle\uparrow|S_x|\downarrow\rangle$$

$$+\ \sin\alpha/2\,\cos\alpha/2\,e^{i\gamma Bt}\langle\downarrow|S_x|\uparrow\rangle + \sin^2\alpha/2\,\langle\downarrow|S_x|\downarrow\rangle$$

$$= 2\cos\alpha/2\,\sin\alpha/2\,\cos\gamma Bt\cdot\frac{\hbar}{2} = \frac{\hbar}{2}\sin\alpha\cos\gamma Bt.$$

$$\langle S_y\rangle = \left(\cos\alpha/2\,e^{-i\gamma Bt/2}\langle\uparrow| + \sin\alpha/2\,e^{i\gamma Bt/2}\langle\downarrow|\right)\cdot$$

$$S_y\left(\cos\alpha/2\,e^{i\gamma Bt/2}|\uparrow\rangle + \sin\alpha/2\,e^{-i\gamma Bt/2}|\downarrow\rangle\right) =$$

$$\cos^2\alpha/2\,\langle\uparrow|\cancelto{0}{S_y}|\uparrow\rangle + \cos\alpha/2\,\sin\alpha/2\,e^{-i\gamma Bt}\langle\uparrow|S_y|\downarrow\rangle$$

$$+\ \cos\alpha/2\,\sin\alpha/2\,e^{i\gamma Bt}\langle\downarrow|S_y|\uparrow\rangle + \sin^2\alpha/2\,\langle\downarrow|\cancelto{0}{S_y}|\downarrow\rangle$$

$$= \cos\alpha/2\,\sin\alpha/2\,e^{-i\gamma Bt}(-i) + \cos\alpha/2\,\sin\alpha/2\,e^{i\gamma Bt}(i)$$

$$= -2\cos\alpha/2\,\sin\alpha/2\,\sin\gamma Bt\cdot\hbar/2 = -\frac{\hbar}{2}\sin\alpha\sin\gamma Bt.$$

$$\langle S_z\rangle = \left(\cos\alpha/2\,e^{-i\gamma Bt/2}\langle\uparrow| + \sin\alpha/2\,e^{i\gamma Bt/2}\langle\downarrow|\right)\cdot$$

$$S_z\left(\cos\alpha/2\,e^{i\gamma Bt/2}|\uparrow\rangle + \sin\alpha/2\,e^{-i\gamma Bt/2}|\downarrow\rangle\right) =$$

$$\frac{\hbar}{2}\left(\cos^2\alpha/2 - \sin^2\alpha/2\right) = \frac{\hbar}{2}\cos\alpha.$$

Combining these results, we have

$$\boxed{\langle\vec{S}\rangle = \frac{\hbar}{2}\left(\sin\alpha\cos\gamma Bt\,\hat{x} - \sin\alpha\sin\gamma Bt\,\hat{y} + \cos\alpha\,\hat{z}\right)}$$

This clearly describes an electron precessing around the magnetic field (i.e., $\hat{z}$) at some constant angle α.

$\omega \equiv \gamma B$ is the "Larmor frequency."