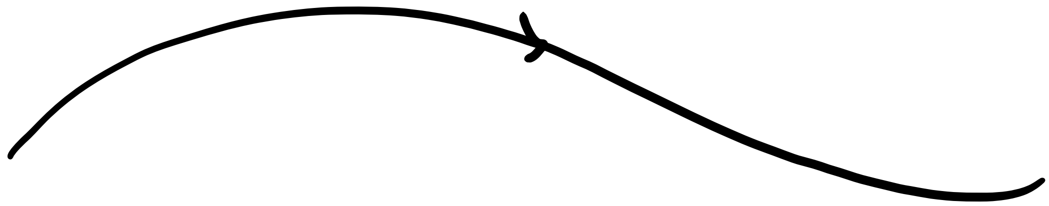
Exercise: We derive the velocity profile and net current for laminar flow in a cylindrical pipe of radius R and length L driven by a pressure difference Δp. We also assume no-slip boundary conditions.

(Source: Katifori – Biological Flow Networks: The Absolute Basics)



The frictional shear force between two layers of fluid in a cylinder is given by $F = -\mu \cdot 2\pi r L \cdot \frac{dv_z}{dr}$, where the fluid flows along the z axis, the layers are concentric cylinders in r, and μ is the viscosity.

The force on a cylindrical shell due to the pressure difference is

$$F = \Delta p \cdot A = \Delta p \cdot 2\pi r \, dr .$$

We assume there are no net forces on our fluid, so the force due to the pressure will balance with the frictional shear forces acting at r (due to inner layer) and $r+dr$ (due to outer layer). The no-slip boundary condition,

$$v(R) = 0$$

ensures that $v(r+dr) < v(r)$, so the frictional force will have opposite signs.

Equilibrium can then be expressed

$$\sum_i F_i = 0 \iff \Delta p \cdot 2\pi r\,dr = \mu L\left(2\pi r\, v'(r) - 2\pi(r+dr)\, v'(r+dr)\right)$$

$$\Rightarrow \Delta p = \mu L\left[\frac{v'(r) - v'(r+dr)}{dr} - \frac{1}{r} v'(r+dr)\right] \approx$$

$$-\mu L\left(v''(r) + \frac{1}{r} v'(r)\right) \Rightarrow$$

$$v''(r) + \frac{1}{r} v'(r) = -\frac{\Delta p}{\mu L}.$$

Solving this differential equation yields the velocity profile in the pipe.

Ansatz: $v(r) = Ar^2 + Br + C$. Plugging in, we get

$$4A + \frac{B}{r} = \frac{-\Delta p}{\mu L}.$$

To enforce that the velocity be finite at the center of the pipe, we require $B = 0$.

Thus, $A = -\dfrac{\Delta p}{4\mu L}$. Next, our no-slip boundary condition requires

that

$$\frac{\Delta p}{4\mu L} R^2 = C \implies$$

$$\boxed{v(r) = \frac{\Delta p R^2}{4\mu L}\left(1 - r^2/R^2\right)}$$

The net flux in the pipe is then given by

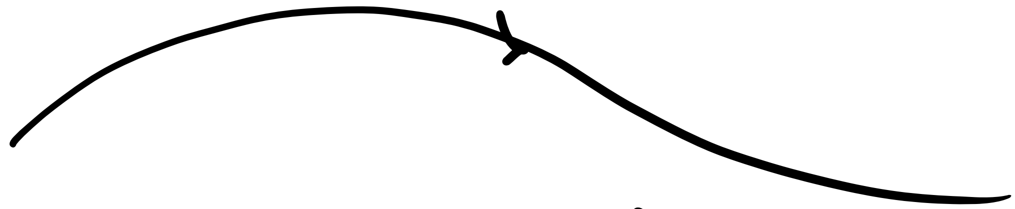
$$Q = \int v\, dA = 2\pi \frac{\Delta p R^2}{4\mu L} \int_0^R \left(1 - r^2/R^2\right) r\, dr$$

$$= \frac{\pi \Delta p R^4}{2\mu L} \int_0^1 (\alpha - \alpha^3)\, d\alpha = \boxed{\frac{\pi \Delta p R^4}{8\mu L}}$$

The "flow conductance" of the pipe is defined to be

$$C \equiv \frac{\pi R^4}{8\mu L} \rightarrow Q = C \Delta p.$$

Thus, we can now clearly see that the flow through the pipe is proportional to the pressure difference along the pipe.



Next, we compute the velocity profile and flux in the case of linear viscosity: $\mu(r) = b_1 r$.

As before, the condition of equilibrium gives us

$$\Delta p \cdot 2\pi r\, dr = \mu(r) L \left[2\pi r\, v'(r) - 2\pi (r+dr)\, v'(r+dr)\right] \Rightarrow$$

$$\Delta p = b_1 r L \left[\frac{v'(r) - v'(r+dr)}{dr} - \frac{1}{r} v'(r+dr)\right] \approx$$

$$-b_1 r L \left[v''(r) + \frac{1}{r} v'(r)\right].$$

The radial velocity profile will thus be governed by

$$r v''(r) + v'(r) = -\frac{\Delta p}{b_1 L}.$$

To solve this equation, once again we make the ansatz

$$v(r) = Ar^2 + Br + C.$$

Plugging in, we get

$$4Ar + B = -\frac{\Delta p}{b_1 L},$$

and setting $r=0 \rightarrow B = -\frac{\Delta p}{b_1 L} \Rightarrow 4Ar = 0 \ \forall r$

$\Rightarrow A = 0.$

Thus, we have $v(r) = C - \frac{\Delta p}{b_1 L} r$. The no-slip BC gives $v(R) = 0 \Leftrightarrow C = \frac{\Delta p}{b_1 L} R \Rightarrow$

$$\boxed{v(r) = \frac{\Delta p R}{b_1 L}(1 - r/R)}$$

So, in contrast to the case of constant viscosity, the velocity now decreases linearly from the center to the edge of the pipe.

The mass flux is once again given by

$$Q = 2\pi \int_0^R r v(r)\, dr = 2\pi \frac{\Delta p R}{b_1 L} \int_0^R r(1 - r/R)\, dr$$

$$= \frac{2\pi \Delta p R^3}{b_1 L} \int_0^1 (\alpha - \alpha^2)\, d\alpha = \boxed{\frac{\pi \Delta p R^3}{3 b_1 L}}$$

In this case, our conductance can be defined $C(R) \equiv \frac{\pi R^3}{3 b_1 L}$,

so that once again we have $Q = C \Delta p$.

Finally, we consider the case where $\mu(r) = b_2 / r$.

Equilibrium gives

$$\Delta p \cdot 2\pi r\, dr = \mu(r) L \left[2\pi r v'(r) - 2\pi (r+dr) v'(r+dr) \right] \Rightarrow$$

$$\Delta p = b_2 L \cdot \frac{1}{r} \left[\frac{v'(r) - v'(r+dr)}{dr} - \frac{1}{r} v'(r+dr) \right] \approx$$

$$- \frac{b_2 L}{r} \left[v''(r) + \frac{1}{r} v'(r) \right] \Rightarrow$$

$$\frac{1}{r}v''(r) + \frac{1}{r^2}v'(r) = -\frac{\Delta p}{b_2 L}.$$

We solve this equation with the ansatz

$$v(r) = Ar^3 + Br^2 + Cr + D.$$

Plugging in, we have

$$6A + \frac{2B}{r} + 3A + \frac{2B}{r} + \frac{C}{r^2} = -\frac{\Delta p}{b_2 L}.$$

For this to hold for $r=0$, we require $B=C=0 \rightarrow$

$$A = -\frac{\Delta p}{9b_2 L}.$$

So now

$$v(r) = -\frac{\Delta p}{9b_2 L}r^3 + D,$$

and the no-slip boundary condition gives $v(R) = 0 \Leftrightarrow$

$$\boxed{v(r) = \frac{\Delta p R^3}{9 b_2 L}\left(1 - r^3/R^3\right)}$$

The mass flux is thus given by

$$Q = 2\pi \int_0^R r\, v(r)\, dr = \frac{2\pi \Delta p R^5}{9 b_2 L} \int_0^1 \alpha(1-\alpha^3)\, d\alpha$$

$$= \boxed{\frac{\pi \Delta p R^5}{15 b_2 L}}.$$

In this case, the conductance is given by $C(R) \equiv \dfrac{\pi R^5}{15 b_2 L} \rightarrow Q = C \Delta p.$