Exercise: We derive the LU decomposition.

(Source: Princeton Companion to Mathematics, Numerical Analysis)

Suppose we wish to solve the matrix equation

$$Ax = b$$

where A is $n \times n$. We can accomplish this with Gaussian elimination, where we transform A into an upper-triangular matrix U through a sequence of row operations (i.e. scalar multiplication of one row, and adding it to another row).

Adding $\gamma \cdot (\text{row } j)$ to row i can be accomplished by left multiplication with $\mathbb{1} + \gamma \delta_{ij}$

For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g + \delta a & h + \delta b & i + \delta c \end{pmatrix}$$

$\mathbb{1} + \delta \cdot \delta_{31}$ added $\delta \cdot$ (row 1) to row 3.

Each such row operation can be represented by a lower triangular matrix M_j. Thus, we have

$$M_k \cdot \dots \cdot M_1 A = U \equiv MA \implies$$

$$\boxed{A = M^{-1} U \equiv LU}$$

As the inverse of a lower triangular matrix is also lower-triangular, we have expressed A as a product of a lower-triangular matrix and an upper triangular matrix.