1D Infinite Square Well

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Exercise:

We solve the quantum mechanical infinite square well in 1D.

Solution:

Consider the potential

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

Outside the well, $\psi(x) = 0$, as it would require an infinite force for the particle to enter that region. Inside the well, the Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \iff \psi'' = -\frac{2mE}{\hbar^2}\psi \equiv -k^2\psi,$$

where $k \equiv \sqrt{2mE}/\hbar$ and $E \geq 0$.

Of course, this is the simple harmonic oscillator equation with solutions

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

The wave function must be continuous at the boundary, which means that $\psi(0) = \psi(a) = 0$.

$$\psi(0) = 0 = B \implies \psi(x) = A\sin(kx).$$
$$\psi(a) = 0 = A\sin(ka) \implies ka = n\pi \iff k_n = \frac{n\pi}{a}, n \in \mathbb{Z}.$$

However, $n = 0$ would make the wave function vanish, and negative values of n just rescale the corresponding positive-n solution by a factor of (-1), so we restrict our attention to $n \in \mathbb{N}$.

We normalize each state.

$$1 = \int_0^a |\psi_n|^2 dx = \int_0^a A^2\sin^2\Big(\frac{n\pi x}{a}\Big)dx = A^2\frac{a}{2} \implies A = \sqrt{\frac{2}{a}}.$$

Combining our results, we thus have the 1D infinite square well spectrum

$$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}, \quad n \in \mathbb{N},}$$

and the energy eigenstates

$$\boxed{\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad n \in \mathbb{N}.}$$