We derive the heat capacity per particle of an ideal gas. We recall that the heat capacity is given by

$$C_V = \frac{\partial \langle E \rangle}{\partial T}\bigg|_V .$$

To find $\langle E \rangle$, we first compute the partition function. The Hamiltonian for the ideal gas in a box of volume V is given by

$$H = \sum_{k=1}^{N} \vec{p}_k^{\,2}/2m .$$

assuming each particle has mass m. It follows that the partition function is given by

$$Z = \frac{1}{N!}\frac{1}{(2\pi\hbar)^{3N}} \int d\Gamma \, e^{-\beta H}$$

where $d\Gamma$ is the phase space differential volume element, and the other coefficients account for the indistinguishability of the particles and the fact that the partition function is dimensionless. Plugging in, we have

$$Z = \frac{1}{N!}\frac{1}{(2\pi\hbar)^{3N}} \int \prod_{k=1}^{N} d^3x_k \, d^3p_k \, e^{-\beta \vec{p}_k^{\,2}/2m} =$$

$$\frac{V^N}{N!}\frac{1}{(2\pi\hbar)^{3N}} \left(4\pi \int_{-\infty}^{\infty} p^2 e^{-\beta p^2/2m} \, dp \right)^N = \frac{V^N}{N!}\frac{1}{(2\pi\hbar)^{3N}} \left(\frac{2\pi m}{\beta} \right)^{3N/2} .$$

Next, we compute

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z = -\frac{\partial}{\partial \beta}\left(\log \beta^{-3N/2} + \log \_\_ \right)$$

$$= \frac{3N}{2\beta} = \frac{3N k_B T}{2}.$$

Finally, we compute the heat capacity

$$\boxed{C_v = \left.\frac{\partial \langle E \rangle}{\partial T}\right|_V = \frac{3N k_B}{2} \Longleftrightarrow \frac{C_v}{N} = \frac{3}{2} k_B}$$