Hard Sphere Collision Time

Matt Kafker

Exercise:

We predict the time of collision between two hard spheres with initial configurations (x_1, v_1) and (x_2, v_2) respectively.

Solution:

We consider two hard spheres of radius R. We wish to find the time t such that $|x_1(t) - x_2(t)| = 2R$. Expanding this out, we have

$$|x_1(t) - x_2(t)| = 2R \implies 4R^2 = (x_1(t) - x_2(t)) \cdot (x_1(t) - x_2(t)) = \\ \Big((x_1 - x_2) + (v_1 - v_2)t\Big) \cdot \Big((x_1 - x_2) + (v_1 - v_2)t\Big) = \\ x_{12}^2 + 2x_{12} \cdot v_{12}t + v_{12}^2 t^2,$$

where we have introduced simplifying notation. We further define $b \equiv x_{12} \cdot v_{12}$, and we arrive at the quadratic equation

$$v_{12}^2 t^2 + 2bt + x_{12}^2 - 4R^2 = 0 \implies \\ t = \frac{-b \pm \sqrt{b^2 - v_{12}^2(x_{12}^2 - 4R^2)}}{v_{12}^2}.$$

We are interested in the soonest collision, so we take the negative root, and we derive

$$\boxed{t_{\text{coll}} = \frac{-b - \sqrt{b^2 - v_{12}^2(x_{12}^2 - 4R^2)}}{v_{12}^2}.}$$

We note that if $b < 0$ or the discriminant is negative, no collision will occur.