We derive Hamilton's equations of motion.

The Hamiltonian is defined using the Lagrangian as follows:

$$H(\vec{q}_1, \dots, \vec{q}_N, \vec{p}_1, \dots, \vec{p}_N, t) = \sum_i \vec{p}_i \cdot \dot{\vec{q}}_i - L(\vec{q}_1, \dots, \vec{q}_N, \dot{\vec{q}}_1, \dots, \dot{\vec{q}}_N, t)$$

We can therefore compute its total differential in two different ways:

$$dH = \sum_i \frac{\partial H}{\partial \vec{q}_i} \cdot d\vec{q}_i + \frac{\partial H}{\partial \vec{p}_i} \cdot d\vec{p}_i + \frac{\partial H}{\partial t} dt =$$

$$\sum_i d(\vec{p}_i \cdot \dot{\vec{q}}_i) - \frac{\partial L}{\partial \vec{q}_i} \cdot d\vec{q}_i - \frac{\partial L}{\partial \dot{\vec{q}}_i} d\dot{\vec{q}}_i - \frac{\partial L}{\partial t} dt \overset{\text{Euler-Lagrange}}{=}$$

$$\sum_i d(\vec{p}_i \cdot \dot{\vec{q}}_i) - \frac{\partial L}{\partial \vec{q}_i} \cdot d\vec{q}_i - \vec{p}_i \, d\dot{\vec{q}}_i - \frac{\partial L}{\partial t} dt$$

$$= \sum_i -\frac{\partial L}{\partial \vec{q}_i} \cdot d\vec{q}_i + \dot{\vec{q}}_i \cdot d\vec{p}_i - \frac{\partial L}{\partial t} dt .$$

As these two expressions for dH are independent, we can compare them term by term:

$$\frac{\partial H}{\partial \vec{q}_i} = -\frac{\partial L}{\partial \vec{q}_i} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}_i} = -\dot{\vec{p}}_i ,$$

$$\frac{\partial H}{\partial \vec{p}_i} = \dot{\vec{q}}_i \quad ,$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad .$$

Thus, Hamilton's equations of motion are given by

$$\boxed{\frac{\partial H}{\partial \vec{q}_i} = -\dot{\vec{p}}_i \,, \quad \frac{\partial H}{\partial \vec{p}_i} = \dot{\vec{q}}_i \,, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$