Griffiths Introduction Quantum Mechanics: Solutions

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Chapter 1

Problem 1.1

We consider the distribution of ages

- $N(14) = 1$
- $N(15) = 1$
- $N(16) = 3$
- $N(22) = 2$
- $N(24) = 2$
- $N(25) = 5$

where $N(j)$ is the number of people with age j.

a.) We compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.

$$\langle j^2 \rangle = \sum_j j^2 P(j) = \sum_j j^2 \frac{N(j)}{N} =$$
$$\frac{1}{14}(1 \cdot 14^2 + 1 \cdot 15^2 + 3 \cdot 16^2 + 2 \cdot 22^2 + 2 \cdot 24^2 + 5 \cdot 25^2) \approx \boxed{459.571.}$$

$$\langle j \rangle = \sum_j j P(j) = \sum_j j \frac{N(j)}{N} =$$
$$\frac{1}{14}(1 \cdot 14 + 1 \cdot 15 + 3 \cdot 16 + 2 \cdot 22 + 2 \cdot 24 + 5 \cdot 25) = 21 \implies$$
$$\langle j \rangle^2 = \boxed{441.}$$

b.) We compute the standard deviation of this distribution using $\sigma^2 = \langle \Delta j^2 \rangle$, where $\Delta j = j - \langle j \rangle$.

- $j = 14 \rightarrow \Delta j = -7$

- $j = 15 \to \Delta j = -6$
- $j = 16 \to \Delta j = -5$
- $j = 22 \to \Delta j = 1$
- $j = 24 \to \Delta j = 3$
- $j = 25 \to \Delta j = 4$

Thus, we have

$$\sigma^2 = \langle \Delta j^2 \rangle = \sum_j \Delta j^2 P(j) = \sum_j \Delta j^2 \frac{N(j)}{N} =$$
$$\frac{1}{14}(1 \cdot 49 + 1 \cdot 36 + 3 \cdot 25 + 2 \cdot 1 + 2 \cdot 9 + 5 \cdot 16) \approx 18.5714 \implies$$
$$\sigma \approx \boxed{4.31.}$$

c.) We compare our results with the formula

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = \sqrt{459.571 - 441} \approx \sqrt{18.571} \approx \boxed{4.31.}$$

We have thus shown that the two formulas give equivalent results for this distribution.

Problem 1.2

We consider the probability distribution $\rho(x) = \frac{1}{2\sqrt{hx}}, 0 \leq x \leq h$.

a.) We find the standard deviation of this distribution using $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$.

$$\langle x \rangle = \int x \rho(x) dx = \frac{1}{2\sqrt{h}} \int_0^h \sqrt{x} dx = \frac{h}{3}.$$

$$\langle x^2 \rangle = \int x^2 \rho(x) dx = \frac{1}{2\sqrt{h}} \int_0^h x^{3/2} dx = \frac{h^2}{5}.$$

It follows that

$$\sigma^2 = \frac{h^2}{5} - \frac{h^2}{9} = \frac{4h^2}{45} \implies$$
$$\sigma = \boxed{\frac{2h}{3\sqrt{5}}.}$$

b.) We compute the probability that a photograph chosen at random shows a distance more than one standard deviation away from the average distance.

$P(\text{distance} > 1 \text{ standard deviation from mean}) = 1 - P(\text{distance} < 1 \text{ standard deviation from mean}) =$

$$1 - \int_{\langle x\rangle-\sigma}^{\langle x\rangle+\sigma} \rho(x)dx = \boxed{1 - \sqrt{\frac{2}{15}\left(5-\sqrt{5}\right)} \approx 0.39.}$$

Problem 1.3

To solve this problem, we will need the following integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} dx \quad , \quad \int_{-\infty}^{\infty} xe^{-x^2} dx \quad , \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx.$$

We derive the solution to each of them.

$$\Big(\int_{-\infty}^{\infty} e^{-x^2} dx\Big)^2 = \Big(\int_{-\infty}^{\infty} e^{-x^2} dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2} dy\Big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy =$$
$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta = 2\pi \cdot \frac{1}{2}\int_0^{\infty} e^{-r^2}(2r)dr = \pi\int_0^{\infty} e^{-u}du = \pi \implies$$
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \boxed{\sqrt{\pi}.}$$

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = 0$$

since xe^{-x^2} is an odd function.

$$\int_{\infty}^{\infty} x^2 e^{-x^2} dx = \int_{\infty}^{\infty} \Big(-\frac{\partial}{\partial\alpha} e^{-\alpha x^2} dx\Big)|_{\alpha=1} = -\frac{\partial}{\partial\alpha}\int_{\infty}^{\infty} e^{-\alpha x^2} dx|_{\alpha=1} =$$
$$-\frac{\partial}{\partial\alpha}\frac{1}{\sqrt{\alpha}}\int_{\infty}^{\infty} e^{-(\sqrt{\alpha}x)^2} d(\sqrt{\alpha}x)|_{\alpha=1} = -\frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{\alpha}}|_{\alpha=1} =$$
$$(-\sqrt{\pi})(-\frac{1}{2}\alpha^{-3/2})|_{\alpha=1} = \boxed{\frac{\sqrt{\pi}}{2}.}$$

We are now ready to address the question.

a.) We consider the gaussian distribution $\rho(x) = Ae^{-\lambda(x-a)^2}$, where $A, \lambda, a > 0$. We solve for A using the requirement that $\rho(x)$ be normalized.

$$\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx = 1 = A\int_{-\infty}^{\infty} e^{-(\sqrt{\lambda}(x-a))^2}\frac{1}{\sqrt{\lambda}}d(\sqrt{\lambda}(x-a)) =$$

$$\frac{A}{\sqrt{\lambda}}\int_{-\infty}^{\infty} e^{-u^2}du = A\sqrt{\frac{\pi}{\lambda}} \implies \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

b.) Next, we compute $\langle x\rangle, \langle x^2\rangle,$ and σ.

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)dx = \int_{-\infty}^{\infty} x\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(x-a)^2}dx = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\lambda}}\Big(\sqrt{\lambda}(x-a)+\sqrt{\lambda}a\Big)e^{-\lambda(x-a)^2}dx =$$

$$a\Big(\int_{-\infty}^{\infty}\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(x-a)^2}dx\Big) + \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}\sqrt{\lambda}(x-a)e^{-(\sqrt{\lambda}(x-a))^2}\frac{1}{\lambda}d(\sqrt{\lambda}(x-a)) =$$

$$a + \sqrt{\frac{\lambda}{\pi}}\cdot\frac{1}{\lambda}\int_{-\infty}^{\infty} ue^{-u^2}du \implies \boxed{\langle x\rangle = a.}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho(x)dx = \int_{-\infty}^{\infty} x^2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(x-a)^2}dx.$$

Let $u = \sqrt{\lambda}(x-a)$. Then, $x = \frac{u}{\sqrt{\lambda}} + a, x^2 = \frac{u^2}{\lambda} + 2\frac{a}{\sqrt{\lambda}}u + a^2, dx = du/\sqrt{\lambda}$. Substituting, we have

$$\int_{-\infty}^{\infty} x^2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(x-a)^2}dx = \sqrt{\frac{\lambda}{\pi}}\cdot\frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty}\Big(\frac{u^2}{\lambda}+2\frac{a}{\sqrt{\lambda}}u+a^2\Big)e^{-u^2}du =$$

$$\frac{1}{\sqrt{\pi}}\Big[\frac{1}{\lambda}\Big(\int_{-\infty}^{\infty}u^2e^{-u^2}du\Big) + 2\frac{a}{\sqrt{\lambda}}\Big(\int_{-\infty}^{\infty}ue^{-u^2}du\Big) + a^2\Big(\int_{-\infty}^{\infty}e^{-u^2}du\Big)\Big] \implies$$

$$\boxed{\langle x^2\rangle = \frac{1}{2\lambda} + a^2.}$$

We can now simply compute the standard deviation.

$$\boxed{\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{2\lambda}}.}$$

c.) We sketch a graph of $\rho(x)$.

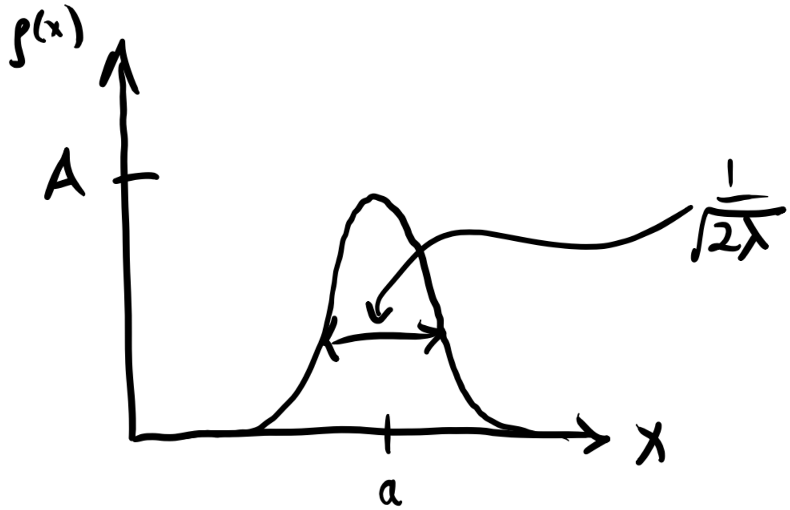

Figure 1: Problem 1.3: Sketch of $\rho(x)$.

Problem 1.4

We consider the wave function

$$\Psi(x,0) = \begin{cases} A\frac{x}{a} & 0 \leq x \leq a \\ A\frac{b-x}{b-a} & a \leq x \leq b \end{cases}$$

a.) First, we normalize the wave function.

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx =$$

$$|A|^2\left(\frac{a}{3} + \frac{b-a}{3}\right) = \frac{b}{3} \implies \boxed{A = \sqrt{\frac{3}{b}}.}$$

b.) Next, we sketch the wave function.

c.) The particle is most likely to be found at the place where $|\Psi(x,0)|^2$ is maximized, which in this case is the same place where $\Psi(x,0)$ is maximum, $\boxed{x = a.}$

d.) The probability of finding the particle to the left of a is given by

$$\int_{-\infty}^{a} |\Psi(x,0)|^2 dx = \frac{3}{b}\int_0^a \frac{x^2}{a^2} dx = \boxed{\frac{a}{b}.}$$

We see that in the limit $b = a$, we get probability 1, and when $b = 2a$, we get probability 1/2, as we expect.

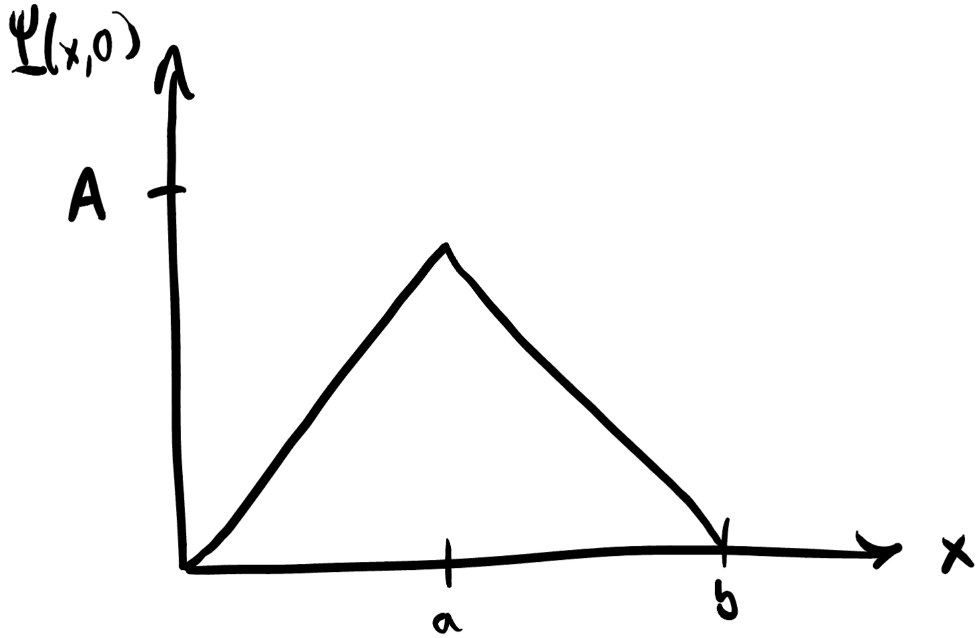

Figure 2: Problem 1.4: Sketch of $\Psi(x,0)$.

e.) Finally, we compute $\langle x \rangle$.

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x,0) x \Psi(x,0) dx = \frac{3}{a^2 b} \int_0^a x^2 \cdot x dx + \frac{3}{b(b-a)^2} \int_a^b (b-x)^2 \cdot x dx = \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b-x)^2 \cdot x dx.$$

We solve the second integral by u-substitution. Let $u = b - x$, and replace $x \mapsto (x-b) + b = -(b-x) + b$. Then, we have

$$\langle x \rangle = \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\int_{b-a}^{0} u^3 du - b \int_{b-a}^{0} u^2 du \right] = \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \Big(-\frac{(b-a)^4}{4} + b \frac{(b-a)^3}{3} \Big) =$$

$$\frac{3a^2}{4b} + \frac{3}{b} \Big(b \frac{b-a}{3} - \frac{(b-a)^2}{4} \Big) = \frac{1}{4b} (3a^2 + 4b^2 - 4ab - 3b^2 + 6ab - 3a^2) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

We consider the wave function $\Psi(x,t) = Ae^{-\lambda |x|} e^{-i\omega t}$. **a.)** First, we normalize Ψ.

$$1 = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 2|A|^2 \int_0^{\infty} e^{-2\lambda x} dx = \frac{|A|^2}{\lambda} \implies$$

$$\boxed{A = \sqrt{\lambda}.}$$

b.) Next, we compute $\langle x\rangle$ and $\langle x^2\rangle$.

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x\left(\lambda e^{-2\lambda|x|}\right)dx = \boxed{0}$$

since the integrand is an odd function.

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x^2\left(\lambda e^{-2\lambda|x|}\right)dx = \int_{0}^{\infty} x^2\left(2\lambda e^{-2\lambda x}\right)dx.$$

Let $\alpha \equiv 2\lambda$. Then, the integral becomes

$$\langle x^2\rangle = \int_0^{\infty} x^2\left(\alpha e^{-\alpha x}\right)dx = \alpha\int_0^{\infty} \frac{\partial^2}{\partial\alpha^2}e^{-\alpha x}dx = \alpha\frac{\partial^2}{\partial\alpha^2}\int_0^{\infty} e^{-\alpha x}dx =$$

$$\alpha\frac{\partial^2}{\partial\alpha^2}\frac{1}{\alpha} = \alpha\frac{2}{\alpha^3} = \frac{2}{\alpha^2} = \boxed{\frac{1}{2\lambda^2}.}$$

c.) We compute the standard deviation of this distribution.

$$\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} = \boxed{\frac{1}{\sqrt{2}\lambda}.}$$

Next, we sketch the probability distribution associated with this wave function.

The probability that the particle will be found more than one standard deviation from the mean is given by

$$\text{prob}(x \notin [\langle x\rangle - \sigma, \langle x\rangle + \sigma]) = 1 - \text{prob}(x \in [\langle x\rangle - \sigma, \langle x\rangle + \sigma]) =$$

$$1 - \int_{-\sigma}^{\sigma} |\Psi(x,t)|^2 = 1 - 2\lambda\int_0^{\sigma} e^{-2\lambda x}dx = 1 - \int_0^{\sigma/2\lambda} e^{-u}du =$$

$$e^{-\sigma/2\lambda} = \boxed{e^{-\sqrt{2}} \approx 0.243.}$$

Problem 1.6

We cannot integrate the equation

$$\frac{d\langle x\rangle}{dt} = \int x\partial_t|\Psi|^2 dx$$

by parts to move the time derivative onto x because we are differentiating with respect to *time*, whereas we are integrating with respect to *space*.

The product rule for differentiation gives us that

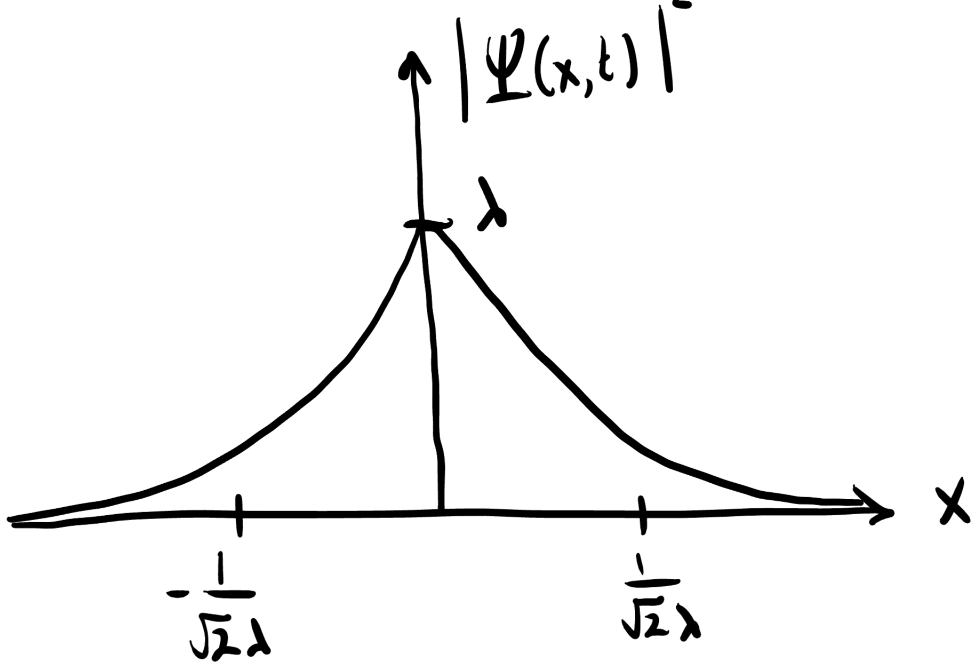

Figure 3: Problem 1.5: Sketch of $|\Psi(x,t)|^2$.

$$\partial_t(x|\Psi|^2) = (\partial_t x)|\Psi|^2 + x\partial_t|\Psi|^2 = x\partial_t|\Psi|^2.$$

We would recover the integration-by-parts identity by integrating both sides of this equation over time, but given that the expectation value involves an integral over space, we are not justified in using integration by parts.

Problem 1.7

We compute $d\langle p\rangle/dt$.

$$\begin{aligned}
\frac{d\langle p\rangle}{dt} &= \frac{d}{dt}\int \Psi^*\Big(-i\hbar\partial_x\Big)\Psi dx \implies \\
\frac{i}{\hbar}\frac{d\langle p\rangle}{dt} &= \int \partial_t\Big(\Psi^*\Psi_x\Big)dx = \int\Big(\dot\Psi^*\Psi_x + \Psi^*\dot\Psi_x\Big)dx = \\
\frac{i}{\hbar}\int\Big[-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi_x + V\Psi^*\Psi_x &- \Big(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xxx} + V_x|\Psi|^2 + V\Psi^*\Psi_x\Big)\Big]dx = \\
\frac{i}{\hbar}\int\Big[-\frac{\hbar^2}{2m}\Big(\Psi^*_{xx}\Psi_x - \Psi^*\Psi_{xxx}\Big) - V_x|\Psi|^2\Big]dx &= -\frac{i}{\hbar}\int V_x|\Psi|^2 dx,
\end{aligned}$$

where the final equality can be demonstrated using integration by parts. (Assuming boundary contributions vanish, each integration by parts moves a derivative from one term to the other, and contributes a factor of -1.)

Thus, we conclude that

$$\boxed{\frac{d\langle p\rangle}{dt} = -\left\langle\frac{\partial V}{\partial x}\right\rangle = \langle F\rangle.}$$

This is known as **Ehrenfest's Theorem**. The two equations

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{\langle p\rangle}{m} \\
\frac{d\langle p\rangle}{dt} &= \langle F\rangle
\end{aligned}$$

tell us that expectation values in quantum mechanics obey classical equations of motion.

Problem 1.8

Suppose $\Psi(x,t)$ obeys the Schrödinger equation, $i\hbar\dot\Psi = -\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi$. Then $\Psi e^{-iV_0t/\hbar}$ obeys the Schrödinger equation with a constant offset in the potential energy.

Proof:

$$i\hbar\partial_t\Big(\Psi e^{-iV_0t/\hbar}\Big) = i\hbar\dot\Psi e^{-iV_0t/\hbar} + i\hbar\Psi e^{-iV_0t/\hbar}(-iV_0/\hbar) = i\hbar\dot\Psi e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar} =$$

$$\left(-\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi\right)e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar} = -\frac{\hbar^2}{2m}\partial_{xx}\left(\Psi e^{-iV_0t/\hbar}\right) + (V+V_0)\Psi e^{-iV_0t/\hbar}.$$

This additional phase factor does not affect any expectation values (excepting operators involving time derivatives) as the complex conjugation will kill any overall phase of the wave function.

Problem 1.9

We consider the wave function $\Psi(x,t) = Ae^{-amx^2/\hbar}e^{-ait}$.

a.) We normalize this wave function.

$$\frac{1}{|A|^2} = \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}dx = \sqrt{\frac{\pi\hbar}{2am}} \implies \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

b.) We determine which potential makes this wave function satisfy the Schrödinger equation.

$$i\hbar\dot{\Psi} = i\hbar\Psi(-ia) = a\hbar\Psi.$$

$$-\frac{\hbar^2}{2m}\Psi_{xx} = -\frac{\hbar^2}{2m}\Psi(-2am/\hbar)(1-2amx^2/\hbar) = a\hbar\Psi(1-2ma^2x^2) \implies$$

$$i\hbar\dot{\Psi} + \frac{\hbar^2}{2m}\Psi_{xx} = V\Psi = 2ma^2x^2\Psi \implies \boxed{V(x) = 2ma^2x^2.}$$

This is a harmonic potential.

c.) Next, we compute $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \text{and}\langle p^2\rangle$.

$$\langle x\rangle = \int_{-\infty}^{\infty} xe^{-2amx^2/\hbar}dx = \boxed{0}$$

since $x|\Psi|^2$ is an odd function.

$$\langle x^2\rangle = \sqrt{\frac{2am}{\pi\hbar}}\int_{-\infty}^{\infty} x^2e^{-2amx^2/\hbar}dx =$$

$$\sqrt{\frac{2am}{\pi\hbar}}\int_{-\infty}^{\infty}\left(-\frac{d}{dk}e^{-kx^2}\right)dx\bigg|_{k=2am/\hbar} = -\sqrt{\frac{2am}{\pi\hbar}}\frac{d}{dk}\int_{-\infty}^{\infty}e^{-kx^2}dx\bigg|_{k=2am/\hbar} =$$

$$-\sqrt{\frac{2am}{\pi\hbar}}\frac{d}{dk}\sqrt{\frac{\pi}{k}}\bigg|_{k=2am/\hbar} = \sqrt{\frac{2am}{\pi\hbar}}\frac{\sqrt{\pi}}{2}k^{-3/2}\bigg|_{k=2am/\hbar} =$$

$$\sqrt{\frac{2am}{\pi\hbar}}\frac{\sqrt{\pi}}{2}\frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

And finally,

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* (-i\hbar\partial_x)^2 \Psi dx = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \Psi_{xx} dx = \hbar^2 \int_{-\infty}^{\infty} \Psi_x^* \Psi_x dx.$$

$$\Psi_x = \Psi(-2amx/\hbar) \implies \Psi_x^* = \Psi^*(-2amx/\hbar) \implies$$

$$\langle p^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} |\Psi|^2 x^2 \left(\frac{2am}{\hbar}\right)^2 dx = 4a^2m^2 \langle x^2 \rangle = 4a^2m^2 \frac{\hbar}{4am} = \boxed{am\hbar.}$$

d.) We verify that the uncertainty principle holds.

$$\sigma_x \sigma_p = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{\hbar}{4am}} \sqrt{am\hbar} = \boxed{\hbar/2.}$$

The uncertainty principle does indeed hold. Furthermore, we see that this wave function actually saturates the uncertainty principle.

Problem 1.10

We consider the first 25 digits of π:

$$3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3.$$

a.) We determine the various probabilities of each digit, if we were to sample this set at random.

- $N(0) = 0 \to P(0) = 0.$
- $N(1) = 2 \to P(1) = 2/25.$
- $N(2) = 3 \to P(2) = 3/25.$
- $N(3) = 5 \to P(3) = 1/5.$
- $N(4) = 3 \to P(4) = 3/25.$
- $N(5) = 3 \to P(5) = 3/25.$
- $N(6) = 3 \to P(6) = 3/25.$
- $N(7) = 1 \to P(7) = 1/25.$
- $N(8) = 2 \to P(8) = 2/25.$
- $N(9) = 3 \to P(9) = 3/25.$

b.) The most probable digit is $\boxed{3,}$ which occurs with probability 1/5.

The median digit can be found by first sorting the list:

$$1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 8, 8, 9, 9, 9.$$

Counting symmetrically towards the middle, we find that the median value is $\boxed{4.}$

The mean value is given by

$$\langle j\rangle = \sum_j jP(j) = 1\cdot\frac{2}{25} + 2\cdot\frac{3}{25} + 3\cdot\frac{5}{25} + 4\cdot\frac{3}{25} + 5\cdot\frac{3}{25} + 6\cdot\frac{3}{25} + 7\cdot\frac{1}{25} + 8\cdot 225 + 9\cdot 325 =$$

$$\frac{2+6+15+12+15+18+7+16+27}{25} = \boxed{\frac{118}{25}\approx 4.72.}$$

c.) Next, we compute the standard deviation. First, this requires computing the second moment of the distribution.

$$\langle j^2\rangle = \sum_j j^2P(j) = 1^2\cdot\frac{2}{25} + 2^2\cdot\frac{3}{25} + 3^2\cdot\frac{5}{25} + 4^2\cdot\frac{3}{25} + 5^2\cdot\frac{3}{25} + 6^2\cdot\frac{3}{25} + 7^2\cdot\frac{1}{25} + 8^2\cdot 225 + 9^2\cdot 325 =$$

$$\frac{2+12+45+48+75+108+49+128+243}{25} = \frac{710}{25}\approx 28.4.$$

Thus, the standard deviation of this list is given by

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} = \sqrt{\frac{710}{25} - \left(\frac{118}{25}\right)^2} = \boxed{\frac{3826}{625}\approx 6.12.}$$

Problem 1.11

a.) Classically, the total energy is formed from the kinetic energy and the potential energy,

$$E = \frac{1}{2}mv(x)^2 + U(x).$$

We may therefore simply solve for $v(x)$:

$$\boxed{v(x) = \sqrt{\frac{2}{m}\Big(E - U(x)\Big)}.}$$

b.) We consider the case of a harmonic oscillator centered on the origin, $U(x) = \frac{1}{2}kx^2$.

In this case, a particle with energy E will oscillate between the turning points $x_0 = \pm\sqrt{\frac{2E}{k}}$. From problem statement, we know that the probability distribution of positions is given by $\rho(x) = 1/v(x)T$, where T is the time it takes the particle to move from one turning point to the other. In this case, T is just half the period of a simple harmonic oscillator:

$$T = \pi\sqrt{m/k}.$$

Thus, we have

$$\rho(x) = \frac{1}{\pi}\sqrt{\frac{k}{m}}\sqrt{\frac{m}{2}}\frac{1}{\sqrt{E-U(x)}} = \frac{1}{\pi}\frac{\sqrt{k/2}}{\sqrt{E-\frac{1}{2}kx^2}} = \boxed{\frac{1}{\pi}\frac{1}{\sqrt{\frac{2E}{k}-x^2}}.}$$

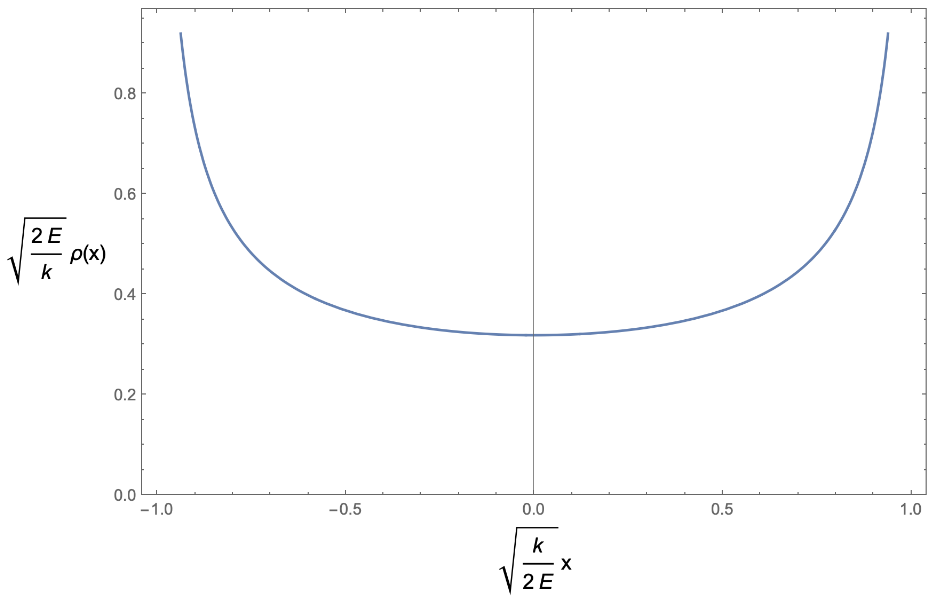

Figure 4: Problem 1.11: Sketch of $\rho(x)$.

We plot the behavior of this density in space in Figure 4.

We can further verify that this probability density is normalized.

$$\int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \rho(x)dx = \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \frac{1}{\pi}\sqrt{\frac{k}{2E}} \frac{1}{\sqrt{1-\left(\sqrt{\frac{k}{2E}}x\right)^2}}dx = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-\alpha^2}}d\alpha = \boxed{1.}$$

c.) Next, we compute $\langle x\rangle, \langle x^2\rangle$, and σ_x for this distribution.

$$\langle x\rangle = 0$$

since $\rho(x)$ is an even function.

$$\langle x^2\rangle = \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} x^2\rho(x)dx = \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \frac{1}{\pi}\sqrt{\frac{k}{2E}} \frac{\left(\sqrt{\frac{k}{2E}}x\right)^2}{\sqrt{1-\left(\sqrt{\frac{k}{2E}}x\right)^2}}dx = \frac{2E}{\pi k}\int_{-1}^{1}\frac{\alpha^2}{\sqrt{1-\alpha^2}}d\alpha = \boxed{\frac{E}{k}.}$$

It follows that $\boxed{\sigma_x = \sqrt{E/k}.}$

Problem 1.12

a.) We calculate the distribution of momenta for the classical simple harmonic oscillator.

We know that the probability distribution must obey $\rho(p)dp = \rho(t)dt$, and assuming we sample at a random time, that gives us

$$\rho(p) = \rho(t)dt/dp = \frac{dt/dp}{T} = \frac{1}{FT},$$

where F is the force and T is the half-period of the simple harmonic oscillator. Substituting the known values of these quantities, we have

$$\rho(p) = -\frac{1}{\pi}\frac{1}{\sqrt{km}}\frac{1}{x}.$$

Energy conservation gives us

$$E = \frac{p(x)^2}{2m} + \frac{1}{2}kx^2 \implies x = \pm\sqrt{\frac{2}{k}\left(E - \frac{p^2}{2m}\right)} \implies$$

$$\boxed{\rho(p) = \frac{1}{\pi}\frac{1}{\sqrt{2mE}}\frac{1}{\sqrt{1-\frac{p^2}{2mE}}}, \text{with } p \in [-\sqrt{2mE}, \sqrt{2mE}].}$$

b.) We calculate the standard deviation of this distribution.

$$\langle p \rangle = \boxed{0,}$$

since $\rho(p)$ is an even function of p.

$$\langle p^2 \rangle = \int_{-\sqrt{2mE}}^{\sqrt{2mE}} p^2\rho(p)dp = 2mE\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{1}{\pi}\frac{1}{\sqrt{2mE}}\frac{\left(\frac{p}{\sqrt{2mE}}\right)^2}{\sqrt{1-\frac{p^2}{2mE}}}dp =$$

$$\frac{2mE}{\pi}\int_{-1}^{1}\frac{\alpha^2}{\sqrt{1-\alpha^2}}d\alpha = \boxed{mE.}$$

Thus, the standard deviation of this distribution is given by

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \boxed{\sqrt{mE}.}$$

c.) Combining our results from this problem and the previous problem, we determine that the classical uncertainty product for this system is given by

$$\sigma_x\sigma_p = \sqrt{\frac{m}{k}}E = \boxed{\frac{E}{\omega}.}$$

For the quantum simple harmonic oscillator, $E \geq \hbar\omega/2$, which means that $\sigma_x\sigma_p \geq \hbar/2$, which is just the Heisenberg uncertainty principle, as we expect.

```
Show[
 Histogram[Cos[Range[0, N[16 π], N[2 π/1000]]], 50, "PDF", PlotRange → {{-1, 1}, {0, 2}},
  Frame → True, FrameLabel → Map[Style[#, Black, 15] &, {"x", "ρ(x)"}]],
 Plot[1/(π Sqrt[1 - x^2]), {x, -1, 1}]]
```

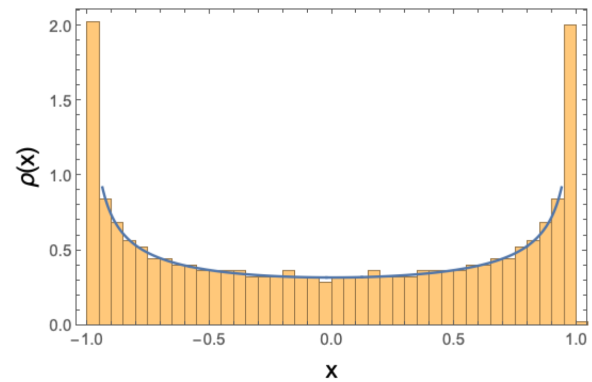

Figure 5: Problem 1.13: Sketch of $\rho(x)$.

Problem 1.13

We generate random samples in time, and produce a histogram of the distribution of positions. We plot the analytical result as well. See figure 5.

Problem 1.14

a.) We demonstrate the conservation of probability current in quantum mechanics.

The probability current is defined as

$$J(x,t) = \frac{i\hbar}{2m}\Big(\frac{\partial \Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial \Psi}{\partial x}\Big).$$

We consider the probability that a particle is found between a and b :

$$P_{ab} = \int_a^b |\Psi|^2 dx.$$

Then, this quantity changes in time as

$$\dot{P}_{ab} = \int_a^b \partial_t |\Psi|^2 dx = \int_a^b \Big(\dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}\Big)dx =$$

$$\int_a^b \frac{i}{\hbar}\Big[\Big(-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi + V|\Psi|^2) - (-\frac{\hbar^2}{2m}\Psi^*\Psi_{xx} + V|\Psi|^2)\Big]dx =$$

$$\frac{i\hbar}{2m}\int_a^b \Big(\Psi^*\Psi_{xx} - \Psi^*_{xx}\Psi\Big)dx = \frac{i\hbar}{2m}\Big(\Psi^*\Psi_x - \Psi^*_x\Psi\Big)\Big|_a^b =$$

$$-J(x,t)\Big|_a^b = J(x,t)\Big|_b^a = \boxed{J(a,t) - J(b,t).}$$

Thus, in quantum mechanics, the change in probability in some region of space equals the net flux of probability current at the boundary.

We find the units of the probability current.

$$[\hbar] = \frac{L^2M}{T}, [\Psi] = \frac{1}{\sqrt{L}}, [\Psi_x] = \frac{1}{L\sqrt{L}} \implies \boxed{[J] = \frac{1}{\text{time}}.}$$

b.) We find the probability current for the wave function $\Psi(x,t) = Ae^{-amx^2/\hbar}e^{-iat}$.

$$\Psi_x = -2amx/\hbar\Psi \implies$$

$$\frac{2m}{i\hbar}J = \Psi^*_x\Psi - \Psi^*\Psi_x = -(2amx/\hbar)|\Psi|^2 + (2amx/\hbar)|\Psi|^2 = \boxed{0.}$$

Problem 1.15

We show that the Schrödinger equation preserves the overlap between two wave functions in time.

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = \int_{-\infty}^{\infty}(\dot{\Psi}_1^*\Psi_2 + \Psi_1^*\dot{\Psi}_2)dx =$$

$$\int_{-\infty}^{\infty}\frac{i}{\hbar}\Big[-\frac{\hbar^2}{2m}\Psi^*_{1,xx}\Psi_2 + V\Psi_1^*\Psi_2 - \Big(-\frac{\hbar^2}{2m}\Psi_1^*\Psi_{2,xx} + V\Psi_1^*\Psi_2\Big)\Big]dx =$$

$$-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Big(\Psi^*_{1,xx}\Psi_2 - \Psi_1^*\Psi_{2,xx}\Big)dx = \boxed{0.}$$

The final equality can be obtained by integrating one of the terms by parts twice.

Problem 1.16

We consider the wave function $\Psi(x,0) = A(a^2 - x^2)$.

a.) First, we normalize the wave function.

$$1 = \int_{-a}^{a}|\Psi|^2 dx = 2A^2\int_0^a (a^2 - x^2)^2 dx = 2A^2\int_0^a (a^4 - 2a^2x^2 + x^4)dx =$$

$$2A^2a^5(1-2/3+1/5) = \frac{2A^2a^5}{15}(15-10+3) = \frac{16A^2a^5}{15} \implies$$
$$\boxed{A = \frac{\sqrt{15}}{4a^{5/2}}.}$$

b.) Next, we compute $\langle x \rangle$. This is trivial, as $|\Psi|^2$ is an even function, so
$$\boxed{\langle x \rangle = 0.}$$

c.) We now compute $\langle p \rangle$.

$$\langle p \rangle = \int_{-a}^{a} \Psi^*(-i\hbar\Psi_x)dx. \qquad \Psi_x = -2Ax \implies$$
$$\langle p \rangle = -i\hbar \int_{-a}^{a} A(a^2-x^2)(-2Ax)dx = \boxed{0}$$
since the integrand is an odd function.

d.) Next, we compute $\langle x^2 \rangle$.
$$\langle x^2 \rangle = \int_{-a}^{a} |\Psi|^2 x^2 dx = \int_{-a}^{a} A^2(a^2-x^2)^2 x^2 dx =$$
$$\int_{-a}^{a} A^2(a^4-2a^2x^2+x^4)x^2dx = \int_{-a}^{a} A^2(a^4x^2-2a^2x^4+x^6)dx =$$
$$2A^2a^7(1/3-2/5+1/7) = \frac{16A^2a^7}{105} = \frac{16a^7}{105}\frac{15}{16a^5} = \boxed{\frac{a^2}{7}.}$$

e.) We compute $\langle p^2 \rangle$.

$$\langle p^2 \rangle = \int_{-a}^{a} \Psi^*(-\hbar^2)\Psi_{xx}dx = -\hbar^2 \int_{-a}^{a} A(a^2-x^2)(-2A)dx = 4A^2\hbar^2 \int_{0}^{a}(a^2-x^2)dx =$$
$$4A^2\hbar^2\frac{2a^3}{3} = \frac{8A^2a^3\hbar^2}{3} = \frac{8a^3\hbar^2}{3}\frac{15}{16a^5} = \boxed{\frac{5\hbar^2}{2a^2}.}$$

f.) Combining our results, we have $\boxed{\sigma_x = a/\sqrt{7}.}$

g.) $\boxed{\sigma_p = \sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$

h.) We verify the uncertainty principle for this wave function:
$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}}\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \hbar\sqrt{\frac{5}{14}} \approx 0.6\hbar > \frac{\hbar}{2}$$
so we verify that the uncertainty principle is satisfied.

Problem 1.17

We consider a simple model of an unstable particle which decays with some lifetime τ.

a.) Suppose $P(t) = \int_{-\infty}^{\infty} |\Psi|^2 dx$. And suppose Ψ is a solution of the Schrödinger equation with potential $V = V_0 - i\Gamma$, where $\Gamma \in \mathbb{R}_{>0}$. Then,

$$\frac{dP(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \left(\dot{\Psi}^* \Psi + \Psi^* \dot{\Psi}\right) dx =$$

$$\frac{i}{\hbar}\int_{-\infty}^{\infty} \Big[\Big(-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi + (V_0 - i\Gamma)|\Psi|^2\Big) - \Big(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xx} + (V_0 + i\Gamma)|\Psi|^2\Big)\Big] dx =$$

$$-\frac{2\Gamma}{\hbar}P(t) + \frac{i}{\hbar}\int_{-\infty}^{\infty} -\frac{\hbar^2}{2m}\Big(\Psi^*_{xx}\Psi - \Psi^*\Psi_{xx}\Big) dx = -\frac{2\Gamma}{\hbar}P(t) \implies$$

$$\frac{dP(t)}{dt} = -\frac{2\Gamma}{\hbar}P(t).$$

b.) Solving this equation, we get

$$\boxed{P(t) = e^{-2\Gamma t/\hbar} \equiv e^{-t/\tau}, \text{ where } \tau = \hbar/2\Gamma.}$$

Thus, we conclude that a solution to the Schrödinger equation with a real potential and a constant imaginary offset describes an unstable particle.

Problem 1.18

A particle must be treated quantum mechanically if its thermal de Broglie wavelength exceeds the characteristic size of the system; that is, if

$$d < \lambda = \frac{h}{\sqrt{3mk_B T}} \iff T < \frac{h^2}{3mk_B d^2}.$$

a.) According to this criterion, unbound electrons in a solid with lattice spacing $d = 0.3$ nm must be treated quantum mechanically for all temperatures

$$\boxed{T < 1.29 \times 10^5 K.}$$

Applying the same criterion to the ion cores of the solid (for example, silicon atoms), we determine that they become quantum mechanical for all temperatures

$$\boxed{T < 2.5K.}$$

Thus, unbound electrons in solids must be treated quantum mechanically, while the ion cores typically do not have to be.

b.) We determine the temperatures for which particles in an ideal gas must be treated quantum mechanically.

According to the ideal gas law, $PV = Nk_BT$, the average interparticle separation is given by

$$\left(\frac{V}{N}\right)^{1/3} = \left(\frac{k_BT}{P}\right)^{1/3}.$$

For the particles to be quantum mechanical, it is required that their thermal de Broglie wavelength exceed this average interparticle separation

$$\left(\frac{k_BT}{P}\right)^{1/3} < \frac{h}{\sqrt{3mk_BT}} \implies T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k^{5/3}} \implies$$

$$\boxed{T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.}$$

For helium at atmospheric pressure, this temperature is about $\boxed{T_{\mathrm{He,1atm}} \approx 2.93K.}$ Whereas for hydrogen in outer space at $3K$, the de Broglie wavelength is about a nanometer. Thus, if the interparticle separation in a gas cloud (for example) is 1cm, the hydrogen will behave classically.