Functional Derivative Cheat Sheet

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Definition:

$$\frac{\delta F}{\delta f(x)} \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(F[f(y) + \epsilon\delta(y-x)] - F[f(y)]\Big)$$

Useful Identities:

$$\frac{\delta}{\delta f(x)} \int f(y)dy = 1$$

$$\frac{\delta}{\delta f(x)} \int f(y)^p \phi(y)dy = pf(x)^{p-1}\phi(x)$$

$$\frac{\delta}{\delta f(x)} \int g(f(y))dy = g'(f(x))$$

$$\frac{\delta}{\delta f(x)} \int g(f'(x))dy = -\frac{d}{dx}g'(f'(x))$$

$$\frac{\delta}{\delta \phi(\vec{x})} \int \Big(\nabla\phi(\vec{y})\Big)^2 d^3y = -2\Delta\phi(\vec{x})$$

$$\frac{\delta}{\delta f(x)} \int G(z,y)f(y)dy = G(z,x)$$

$$\frac{\delta}{\delta \phi(\vec{x})} \int F\Big(\vec{y}, \phi(\vec{y}), \nabla\phi(\vec{y})\Big)d^3y = \frac{\partial}{\partial\phi(\vec{x})}F\Big(\vec{x}, \phi(\vec{x}), \nabla\phi(\vec{x})\Big) - \nabla\cdot\frac{\partial}{\partial(\nabla\phi(\vec{x}))}F\Big(\vec{x}, \phi(\vec{x}), \nabla\phi(\vec{x})\Big)$$

$$\frac{\delta}{\delta f(x)} \int g(y, f(y)) dy = g'(x, f(x))$$

$$\frac{\delta}{\delta f(x)} f(y) = \delta(x - y)$$

$$\frac{\delta}{\delta f(x)} f'(y) = \frac{d}{dx} \delta(y - x)$$

$$\frac{\delta}{\delta f(x)} \int g\Big(y, f(y), f'(y), f''(y)\Big) dy = \frac{\partial g}{\partial f(x)} - \frac{d}{dx} \frac{\partial g}{\partial (f'(x))} + \frac{d^2}{dx^2} \frac{\partial g}{\partial (f''(x))}$$