We compute a number of functional derivatives from Quantum Field Theory for the Gifted Amateur, chapter 1.

We use the definition of the functional derivative for the functional $F[f]$

$$\frac{\delta F}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(F[f(y)+\varepsilon\,\delta(y-x)] - F[f(y)]\Big).$$

1.) Let $I[f] = \int_{-1}^{1} f(x)\,dx$

Then

$$\frac{\delta I}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(I[f(y)+\varepsilon\,\delta(y-x)] - I[f(y)]\Big)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-1}^{1} \big[f(y) + \varepsilon\,\delta(y-x) - f(y)\big]\,dy =$$

$$\int_{-1}^{1} \delta(y-x)\,dy = \boxed{\begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise.} \end{cases} = \frac{\delta}{\delta f(x)} \int_{-1}^{1} f(x)\,dx}$$

This result is analogous to $\frac{d}{dx}(x) = 1$.

2.1

Let $J[f(y)] = \int f(y)^p \phi(y)\,dy$.

Then,

$$\frac{\delta J}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J[f(y) + \varepsilon\delta(y-x)] - J[f(y)] \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[(f(y) + \varepsilon\delta(y-x))^p - f(y)^p \right] \phi(y)\,dx.$$

We Taylor expand x^p near $x = f(y)$.

$$x^p \approx f(y)^p + \frac{d}{dx} x^p \Big|_{x=f(y)} \cdot (x - f(y)).$$

Let $x - f(y) = \varepsilon\delta(y-x)$. Then, we have

$$(f(y) + \varepsilon\delta(y-x))^p \approx f(y)^p + p\,f(y)^{p-1} \cdot \varepsilon\delta(y-x)$$

$$\Rightarrow \frac{\delta J}{\delta f(x)} \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[f(y)^p + p\,f(y)^{p-1} \varepsilon\delta(y-x) - f(y)^p \right] \phi(y)\,dy$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \varepsilon \cdot p\,f(y)^{p-1} \delta(y-x) \phi(y)\,dy =$$

$$\boxed{p\,f(x)^{p-1} \phi(x) = \frac{\delta}{\delta f(x)} \int f(y)^p \phi(y)\,dy}$$

This result is analogous to $\frac{d}{dx}(ax^p) = apx^{p-1}$.

3.1

Let $H[f] = \int g(f(y))dy$.

Then,

$$\frac{\delta H}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(H[f(y) + \varepsilon\delta(y-x)] - H[f(y)] \Big) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \Big[g(f(y) + \varepsilon\delta(y-x)) - g(f(y)) \Big] dy.$$

We Taylor expand $g(x)$ near $x = f(y)$.

$$g(x) \approx g(f(y)) + \frac{dg(x)}{dx}\bigg|_{x=f(y)} \cdot (x - f(y)).$$

Let $x - f(y) = \varepsilon\delta(y-x)$. Then, we have

$$g(f(y) + \varepsilon\delta(y-x)) \approx g(f(y)) + \frac{dg(f(y))}{df(y)} \cdot \varepsilon\delta(y-x)$$

$$\Rightarrow \frac{\delta H[f]}{\delta f(x)} \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \Big[g(f(y)) + \frac{dg(f)}{df} \cdot \varepsilon\delta(y-x) - g(f(y)) \Big] dy$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \varepsilon \frac{dg(f)}{df} \delta(y-x) dy = \boxed{\frac{dg(f(x))}{df(x)} = \frac{\delta}{\delta f(x)} \int g(f(x)) dx}$$

This is analogous to the general statement $\frac{d}{dx} g(x) = \frac{dg}{dx}$.

4.)

Suppose $\bar{V}[x(t')] = \frac{1}{T} \int_0^T V[x(t')] dt'$. Then, using our previous result, we have

$$\frac{\delta \bar{V}}{\delta x(t)} = \frac{d}{dx(t)} \frac{V(x(t))}{T} = \frac{1}{T} V'(x(t)).$$

The functional $\bar{V}[x(t)]$ is, of course, the time-average of the potential energy for a particle evolving in time in one spatial dimension whose trajectory is given by $x(t)$.

5.)

Suppose $J[f(y)] = \int g(f'(y)) dy$. Then

$$\frac{\delta J}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J[f + \varepsilon \delta(y-x)] - J[f] \right) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[g\left(\frac{d}{dy} (f(y) + \varepsilon \delta(y-x)) \right) - g(f'(y)) \right] dy$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[g(f'(y) + \varepsilon \delta'(y-x)) - g(f'(y)) \right] dy.$$

We Taylor expand $g(x)$ near $x = f'(y)$.

$$g(x) \approx g(f'(y)) + \left.\frac{dg(x)}{dx}\right|_{x=f'(y)} \cdot (x - f'(y)).$$

Let $x - f'(y) = \varepsilon\,\delta'(y-x)$. Then, we have

$$g(f'(y) + \varepsilon\,\delta'(y-x)) \approx g(f'(y)) + \frac{dg(f'(y))}{df'(y)} \cdot \varepsilon\,\delta'(y-x)$$

$$\Rightarrow \frac{\delta J}{\delta f(x)} \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[g(f'(y)) + \varepsilon \frac{dg(f'(y))}{df'(y)} \delta'(y-x) - g(f'(y)) \right] dy$$

$$= \int \frac{dg(f'(y))}{df'(y)} \delta'(y-x)\,dy \overset{\text{IBP}}{=} -\int \left(\frac{d}{dy} \frac{dg(f'(y))}{df'(y)} \right) \delta(y-x)\,dy$$

$$= \boxed{-\frac{d}{dx} \frac{dg(f'(x))}{df'(x)} = \frac{\delta}{\delta f(x)} \int g(f'(y))\,dy}$$

This expression is familiar from the Euler-Lagrange equations.

6.) Let $F[\phi] = \int \left(\frac{\partial \phi}{\partial y}\right)^2 dy$. Then,

$$\frac{\delta F}{\delta \phi(x)} = -\frac{\partial}{\partial x}\left(2\frac{\partial \phi}{\partial x}\right) = -2\frac{\partial^2 \phi}{\partial x^2}$$ from our previous result.

7.1

Let $I[\phi] = \int d^3x\,(\nabla\phi)^2$.

Then,

$$\frac{\delta I}{\delta\phi(\vec{x})} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(I[\phi(\vec{y}) + \varepsilon\,\delta^3(\vec{y}-\vec{x})] - I[\phi[\vec{y}]]\right)$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[\left(\nabla(\phi(\vec{y}) + \varepsilon\,\delta^3(\vec{y}-\vec{x}))\right)^2 - (\nabla\phi(\vec{y}))^2\right] d^3y$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[(\nabla\phi(\vec{y}) + \varepsilon\nabla\delta^3(\vec{y}-\vec{x}))^2 - (\nabla\phi(\vec{y}))^2\right] d^3y \approx$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[(\nabla\phi(\vec{y}))^2 + 2\varepsilon\nabla\phi(\vec{y})\cdot\nabla\delta^3(\vec{y}-\vec{x}) - (\nabla\phi(\vec{y}))^2\right] d^3y =$$

$$2\int\nabla\phi(\vec{y})\cdot\nabla\delta^3(\vec{y}-\vec{x})\,d^3y.$$

We now remark that $\nabla(A(\vec{x})B(\vec{x})) = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}(A(\vec{x})B(\vec{x}))$

$$= \sum_i \hat{e}_i\left(\frac{\partial A}{\partial x_i}B(\vec{x}) + A(\vec{x})\frac{\partial B}{\partial x_i}\right) =$$

$$B(\vec{x})\sum_i \hat{e}_i\frac{\partial A}{\partial x_i} + A(\vec{x})\sum_i \hat{e}_i\frac{\partial B}{\partial x_i} = B\nabla A + A\nabla B.$$

We use this result to integrate by parts:

$$\frac{\delta I}{\delta\phi(\vec{x})} = -2\int \nabla\cdot\nabla\phi(\vec{y})\, \delta^3(\vec{y}-\vec{x})\, d^3y = -2\nabla^2\phi$$

$$\Longleftrightarrow \boxed{\frac{\delta}{\delta\phi(x)}\int(\nabla\phi(y))^2 dy = -2\nabla^2\phi(\vec{x})}$$

8.1

Let $I[f] = \int F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\, d^3y$

Then

$$\frac{\delta I}{\delta f(x)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(I[f(\vec{y}) + \varepsilon\delta^3(\vec{y}-\vec{x})] - I[f(\vec{y})]\right)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int F(\vec{y}, f(\vec{y}) + \varepsilon\delta^3(\vec{y}-\vec{x}), \nabla f(\vec{y}) + \varepsilon\nabla\delta^3(\vec{y}-\vec{x}))\, dy$$

We now Taylor expand $F(\vec{y}, z, \vec{w})$ near $z = f(\vec{y})$, $\vec{w} = \nabla f(\vec{y})$.

The multivariate Taylor series is given by

$$F(\vec{y}, z, \vec{w}) \approx F(\vec{y}, f(\vec{y}), \nabla f(\vec{y})) +$$

$$\left(\frac{\partial}{\partial z} F(\vec{y}, z, \vec{w})\Big|_{\substack{z=f(\vec{y}),\\ \vec{w}=\nabla f(\vec{y})}}\right)(z - f(\vec{y})) +$$

$$\left(\nabla_{\vec{w}} F(\vec{y}, z, \vec{w})\Big|_{\substack{z=f(\vec{y}),\\ \vec{w}=\nabla f(\vec{y})}}\right)(\vec{w} - \nabla f(\vec{y})).$$

Let $z - f(\vec{y}) = \varepsilon\,\delta^3(\vec{y}-\vec{x})$, $\vec{w} - \nabla f(\vec{y}) = \varepsilon\,\nabla\delta^3(\vec{y}-\vec{x})$.

Then,

$$F(\vec{y}, f(\vec{y}) + \varepsilon\,\delta^3(\vec{y}-\vec{x}), \nabla f(\vec{y}) + \varepsilon\,\nabla\delta^3(\vec{y}-\vec{x})) \approx$$

$$F(\vec{y}, f(\vec{y}), \nabla f(\vec{y})) + \left(\frac{\partial}{\partial f(\vec{y})} F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\right)\varepsilon\,\delta^3(\vec{y}-\vec{x})$$

$$+ \left(\frac{\partial}{\partial(\nabla f(\vec{y}))} F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\right)\varepsilon\,\nabla\delta^3(\vec{y}-\vec{x}) \Longrightarrow$$

$$\frac{\delta I}{\delta f(\vec{x})} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\Big[F(\vec{y}, f(\vec{y}), \nabla f(\vec{y})) +$$

$$\left(\frac{\partial}{\partial f(\vec{y})} F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\right)\varepsilon\,\delta^3(\vec{y}-\vec{x}) +$$

$$\left(\frac{\partial}{\partial(\nabla f(\vec{y}))} F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\right)\varepsilon\,\nabla\delta^3(\vec{y}-\vec{x}) - F(\vec{y}, f(\vec{y}), \nabla f(\vec{y}))\Big]\,d^3y$$

$$=$$

$$\int\Big[\delta^3(\vec{y}-\vec{x})\left(\frac{\partial}{\partial f(\vec{y})}F(\vec{y},f(\vec{y}),\nabla f(\vec{y}))\right) +$$

$$\nabla\delta^3(\vec{y}-\vec{x})\left(\frac{\partial}{\partial(\nabla f(\vec{y}))}F(\vec{y},f(\vec{y}),\nabla f(\vec{y}))\right)\Big]d^3y \overset{\text{Integration by parts}}{=}$$

$$\int\Big[\delta^3(\vec{y}-\vec{x})\left(\frac{\partial}{\partial f(\vec{y})}F(\vec{y},f(\vec{y}),\nabla f(\vec{y}))\right) -$$

$$\delta^3(\vec{y}-\vec{x})\,\nabla\cdot\left(\frac{\partial}{\partial(\nabla f(\vec{y}))}F(\vec{y},f(\vec{y}),\nabla f(\vec{y}))\right)\Big]d^3y =$$

$$\boxed{\frac{\partial}{\partial f(\vec{x})}F(\vec{x},f(\vec{x}),\nabla f(\vec{x})) - \nabla\cdot\frac{\partial}{\partial(\nabla f(\vec{x}))}F(\vec{x},f(\vec{x}),\nabla f(\vec{x})) = \frac{\delta}{\delta f(\vec{x})}\int F(\vec{y},f(\vec{y}),\nabla f(\vec{y}))\,d^3y}$$

Or, written a bit more simply,

$$\frac{\delta I[f]}{\delta f(\vec{x})} = \frac{\partial F}{\partial f} - \nabla\cdot\frac{\partial F}{\partial(\nabla f)},$$

which is the Euler-Lagrange equation for one dependent variable ($f(\vec{x})$) and three independent variables, $\vec{x}$.

9.

Next, we consider the time-averaged kinetic energy for a 1D particle:

$$\bar{T}[x] = \frac{1}{\tau}\int_0^{\tau} \frac{1}{2} m\, \dot{x}(t)^2\, dt$$

Then, $\frac{\delta \bar{T}[x]}{\delta x(t)}$ takes the form $\frac{\delta}{\delta f(x)} \int g(f'(y))\, dy =$

$$-\frac{d}{dx}\frac{dg(f'(y))}{df'(y)} \Rightarrow \frac{\delta \bar{T}[x]}{\delta x(t)} = -\frac{d}{dt}\frac{m\dot{x}(t)}{\tau} = \frac{-m\ddot{x}(t)}{\tau}.$$

10. We can now derive the principle of least action from Newton's Laws of Motion.

According to Newtonian mechanics,

$$m\ddot{x} = -\frac{dV}{dx} \Longleftrightarrow \frac{\delta \bar{T}}{\delta x(t)} = \frac{\delta \bar{V}}{\delta x(t)} \Longleftrightarrow$$

$$\frac{\delta}{\delta x(t)}(\bar{T}-\bar{V}) = 0.$$ So, the quantity

$$\frac{1}{\tau}\int_0^{\tau}\left(\frac{1}{2} m\, \dot{x}(t)^2 - V(x(t))\right) dt$$ is stationary in classical

mechanics, which trivially implies

$$\int_0^T \left(\frac{1}{2}m\dot{x}(t)^2 - V(x(t))\right) dt \quad \text{is stationary.}$$

We define $L = \frac{1}{2}m\dot{x}(t)^2 - V(x(t))$ to be the Lagrangian, and $S = \int_0^T L\,dt$ to be the action, and we have $\frac{\delta S}{\delta x(t)} = 0$.

Classical particle trajectories thus make the action stationary. $\boxed{/\!/}$

11.| The principle of least action implies the Euler-Lagrange equations of motion:

Using rule 8, we have

$$\frac{\delta S}{\delta x(t)} = \frac{\partial L}{\partial x(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}(t)} = 0 \quad .$$

12.| (Example 1.4)

We consider waves on a string of mass m and length l. Let $\rho = m/l$ be the mass density. We call the tension T and the displacement from equilibrium $\psi(x,t)$.

The kinetic and potential energies are given by

$$T = \frac{1}{2}\int_0^l \rho\, \dot{\psi}(x,t)^2\, dx, \quad V = \frac{1}{2}\int_0^l \tau\, \psi'(x,t)^2 .$$

This is actually a Lagrangian density, so it requires a spatial integration.
We have seen multiple independent variables in example 8.

Then, the action is given by

$$S[\psi] = \int_0^T \int_0^l \left(\frac{1}{2}\rho\, \dot{\psi}(x,t)^2 - \frac{1}{2}\tau\, \psi'(x,t)^2\right) dx\, dt$$

We now extremize the action. Using our results for derivatives appearing in the functional, we have

$$\frac{\delta S}{\delta \psi(x,t)} = 0 = -\frac{d}{dt}\rho\dot{\psi} + \frac{d}{dx}\tau\psi' \iff$$

$$\boxed{\frac{\partial^2}{\partial t^2}\psi = \frac{\tau}{\rho}\frac{\partial^2 \psi}{\partial x^2}}$$

This is the wave equation. $v = \sqrt{\tau/\rho}$ will be the wave speed.

13.1

In fact, I still don't feel comfortable with the differentiation with respect to two independent variables that we just used, so let's be overly cautious and do it again.

$$\text{Let } S[\psi] = \int L\left(y, t', \psi(y,t'), \dot{\psi}(y,t'), \psi'(y,t')\right) dy\, dt'$$

Then

$$\frac{\delta S}{\delta \Psi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(S[\Psi(y,t') + \varepsilon\, \delta(y-x)\,\delta(t'-t)] - S[\Psi(y,t')]\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iint \Big[L\big(y, t', \Psi(y,t') + \varepsilon\, \delta(y-x)\,\delta(t'-t),$$
$$\dot{\Psi}(y,t') + \varepsilon\, \delta(y-x)\,\dot{\delta}(t'-t),$$
$$\Psi'(y,t') + \varepsilon\, \delta'(y-x)\,\delta(t'-t)\big) -$$
$$L(y,t',\Psi,\dot{\Psi},\Psi')\Big]\, dy\, dt \, .$$

We Taylor expand $L(y,t',a,b,c)$ near $(y,t',\Psi(y,t'),\dot{\Psi}(y,t'),\Psi'(y,t'))$:

$$L(y,t',a,b,c) \approx L(y,t',\Psi(y,t'),\dot{\Psi}(y,t'),\Psi'(y,t')) +$$
$$\left(\frac{\partial L}{\partial a}\bigg|_{(y,t',\Psi(y,t'),\dot{\Psi}(y,t'),\Psi'(y,t'))}\right)(a - \Psi(y,t')) +$$
$$\left(\frac{\partial L}{\partial b}\bigg|_{(y,t',\Psi(y,t'),\dot{\Psi}(y,t'),\Psi'(y,t'))}\right)(b - \dot{\Psi}(y,t'))$$
$$\left(\frac{\partial L}{\partial c}\bigg|_{(y,t',\Psi(y,t'),\dot{\Psi}(y,t'),\Psi'(y,t'))}\right)(c - \Psi'(y,t')) .$$

Let $a - \Psi(y,t') = \varepsilon\, \delta(y-x)\,\delta(t'-t)$,
$b - \dot{\Psi}(y,t') = \varepsilon\, \delta(y-x)\,\dot{\delta}(t'-t)$,
$c - \Psi'(y,t') = \varepsilon\, \delta'(y-x)\,\delta(t'-t)$.

Then,

$$\frac{\delta S}{\delta \psi(x,t)} \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iint \Big[L(y,t',\psi(y,t'),\dot{\psi}(y,t'),\psi'(y,t')) +$$

$$\frac{\partial L}{\partial \psi(y,t')} \cdot \varepsilon\, \delta(y-x)\delta(t'-t) + \frac{\partial L}{\partial \dot{\psi}} \cdot \varepsilon\, \delta(y-x)\, \dot{\delta}(t'-t) +$$

$$\frac{\partial L}{\partial \psi'} \varepsilon\, \delta'(y-x)\, \delta(t'-t) - L(y,t',\psi(y,t'),\dot{\psi}(y,t'),\psi'(y,t'))\Big] dy\, dt \overset{\text{IBP}}{=}$$

$$\iint \Big[\frac{\partial L}{\partial \psi} \delta(y-x)\delta(t'-t) - \frac{d}{dt'} \frac{\partial L}{\partial \dot{\psi}} \delta(y-x)\delta(t'-t) -$$

$$\frac{d}{dy}\frac{\partial L}{\partial \psi'} \delta(y-x)\delta(t'-t)\Big] dy\, dt = \boxed{\frac{\partial L}{\partial \psi(x,t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}(x,t)} - \frac{d}{dx}\frac{\partial L}{\partial \psi'(x,t)}}.$$

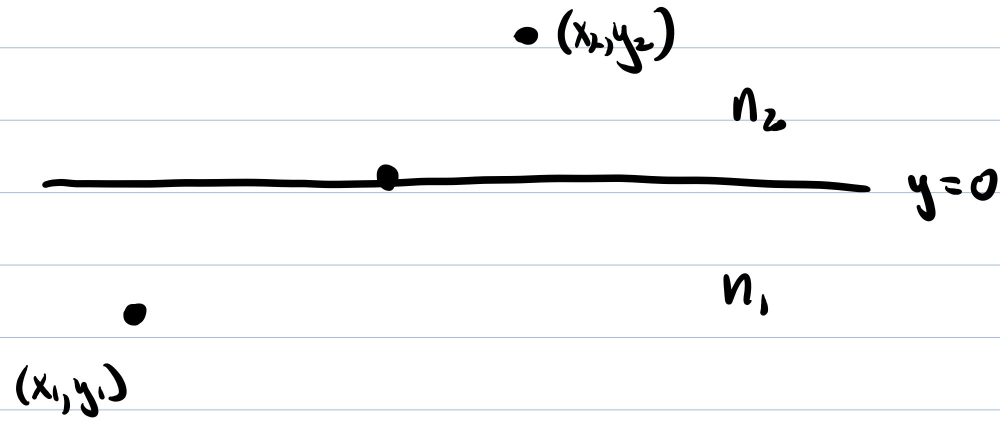
So, indeed, we differentiated correctly.

End of Chapter Problems

1.11

We derive Snell's law using the principle of least time.

We wish to find the trajectory which minimizes the time to go from (x_1, y_1) to (x_2, y_2) where there are two regions with two different indices of refraction:



The time can be expressed as an integral

$$T = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \frac{n_1}{c} \int_{(x_1,y_1)}^{(x^*,0)} ds + \frac{n_2}{c} \int_{(x^*,0)}^{(x_2,y_2)} ds$$

We also recall that $v = c/n$.

We first prove that the least-time trajectory in either medium is a straight line.

Let $f_1(x)$ be the curve that the particle follows from (x_1, y_1) to $(x^*, 0)$, such that $f_1(x_1) = y_1$, $f_1(x^*) = 0$

$$T[f_1] = \frac{n_1}{c}\int_{(x_1,y_1)}^{(x^*,0)} ds\,(f_1) = \frac{n_1}{c}\int_{x_1}^{x^*} \sqrt{1+f_1'(x')^2}\,dx' \Rightarrow$$

$$\frac{\delta T}{\delta f_1(\tilde{x})} = \frac{-d}{dx}\cdot\frac{f_1'(x)}{\sqrt{1+f_1'(x)^2}} = 0 \Rightarrow \frac{f_1'(x)}{\sqrt{1+f_1'(x)^2}} = \text{const} \equiv k$$

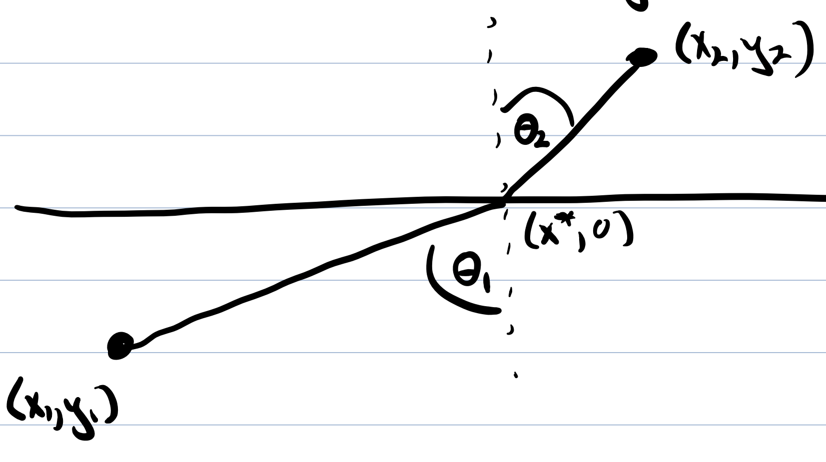
$$\Rightarrow [f_1'(x)]^2 = k^2(1+f_1'(x))^2 = k^2 + k^2 f_1'(x)^2 \Rightarrow$$

$$f_1'(x)^2(1-k^2) = k^2 \Rightarrow f_1'(x) = \sqrt{\frac{k^2}{1-k^2}} = \text{constant.}$$

Thus $f_1'(x) = \text{const} \Rightarrow f_1(x)$ is a straight line!

The same argument implies that $f_2(x)$ is a straight line.

We are now in a position to derive Snell's law. We wish to determine the position of x^* which minimizes the light's travel time.



$$\sin\theta_2 = \frac{x_2 - x^*}{\sqrt{(x_2-x^*)^2+y_2^2}}$$

$$\sin\theta_1 = \frac{x^* - x_1}{\sqrt{(x_1-x^*)^2+y_1^2}}$$

The time the light will travel is now given by

$$T = \frac{n_1}{c}\sqrt{(x_1 - x^*)^2 + y_1^2} + \frac{n_2}{c}\sqrt{(x_2 - x^*)^2 + y_2^2}$$

We find the minimum time by differentiating with respect to x^*:

$$\frac{dT}{dx^*} = 0 = \frac{n_1}{c} \cdot \frac{-(x_1 - x^*)}{\sqrt{(x_1 - x^*)^2 + y_1^2}} + \frac{n_2}{c} \frac{-(x_2 - x^*)}{\sqrt{(x_2 - x^*)^2 + y_2^2}} \Rightarrow$$

$$\frac{n_1 (x^* - x_1)}{\sqrt{(x_1 - x^*)^2 + y_1^2}} = \frac{n_2 (x_2 - x^*)}{\sqrt{(x_2 - x^*)^2 + y_2^2}}$$

However, we can see from the diagram that this expression is equivalent to

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

This is Snell's law.

1.21

Let $H[f] = \int G(x,y) f(y) dy$. We compute $\delta H[f] / \delta f(z)$.

$$\frac{\delta H}{\delta f(z)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(H[f(y) + \varepsilon \delta(y-z)] - H[f(y)] \right) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[G(x,y)(f(y) + \varepsilon \delta(y-z)) - G(x,y) f(y) \right] dy =$$

$$\int G(x,y) \delta(y-z) dy = \boxed{G(x,z) = \frac{\delta}{\delta f(z)} \int G(x,y) f(y) dy}$$

Note: this is analogous to $\frac{\partial}{\partial f_z}(G_{xy} f_y)$, where G_{xy} is a matrix and f_y is a vector.

Let $I[f] = \int_{-1}^{1} f(x) dx$. We compute $\frac{\delta^2 I[f^3]}{\delta f(x_0) \delta f(x_1)}$

$$\frac{\delta I[f^3]}{\delta f(x_1)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(I[(f(x) + \varepsilon \delta(x - x_1))^3] - I[f^3(x)] \right)$$

$$\approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_{-1}^{1} [f(x)^3 + 3 \varepsilon f(x)^2 \delta(x - x_1) - f(x)^3] dx \right)$$

keeping terms upto first order in ε. This simplifies to

$$\int_{-1}^{1} 3f(x)^2 \delta(x-x_1)\, dx = 3f(x_1)^2 .$$

It follows that

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\, \delta f(x_1)} = \frac{\delta}{\delta f(x_0)} \left(3f(x_1)^2\right) =$$

$$\frac{\delta}{\delta f(x_0)} 3\int_{-1}^{1} f(x)^2 \delta(x-x_1) dx = \boxed{6 f(x_0)\, \delta(x_0 - x_1)}$$

We observe that this has similar structure to $\frac{\partial^2}{\partial x \partial y} x^3 = 6\, \delta_{xy}$

or something like that.

Finally, let $J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 dy$. We compute $\frac{\delta J[f]}{\delta f(x)}$

$$\frac{\delta}{\delta f(x)} \int \left(\frac{\partial f}{\partial y}\right)^2 dy = -\frac{d}{dx} \cdot 2\frac{\partial f}{\partial x} = \boxed{-2\frac{\partial^2 f}{\partial x^2}}$$

This case was already considered in the chapter.

1.31

Let $G[f] = \int g(y, f(y))dy$. Then,

$$\frac{\delta G}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(G[f(y) + \varepsilon\delta(y-x)] - G[f(y)]\right) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int \left[g(y, f(y) + \varepsilon\delta(y-x)) - g(y, f(y))\right]dy.$$

We Taylor expand $g(y, z)$ near $z = f(y)$.

$$g(y, z) \approx g(y, f(y)) + \left(\frac{\partial g}{\partial z}\bigg|_{y, z=f(y)}\right)(z - f(y)).$$

Let $z - f(y) = \varepsilon\delta(y-x)$. Then, we have

$$g(y, f(y) + \varepsilon\delta(y-x)) \approx g(y, f(y)) + \frac{\partial g}{\partial f(y)}\varepsilon\delta(y-x)$$

$$\Rightarrow \frac{\delta G}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int \left[g(y, f(y)) + \varepsilon\frac{\partial g}{\partial f(y)}\delta(y-x) - g(y, f(y))\right]dy$$

$$= \int \frac{\partial g(y, f(y))}{\partial f(y)}\delta(y-x)\,dy = \boxed{\frac{\partial g(x, f(x))}{\partial f(x)} = \frac{\delta}{\delta f(x)}\int g(y, f(y))dy}$$

Let $H[f] = \int g(y, f, f')dy$. Then,

$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(H[f(y)+\varepsilon\,\delta(y-x)] - H[f] \right) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \Big[g(y, f(y)+\varepsilon\,\delta(y-x), f'(y)+\varepsilon\,\delta'(y-x)) - g(y, f(y), f'(y)) \Big] dy$$

We Taylor expand $g(y,z,w)$ near $z=f(y), w=f'(y)$.

$$g(y,z,w) \approx g(y,f(y),f'(y)) + \left(\frac{\partial g}{\partial z}\bigg|_{(y,f(y)),f'(y)} \right)(z-f(y))$$

$$+ \frac{\partial g}{\partial w}\bigg|_{(y,f(y)),f'(y)} (w-f'(y)).$$

Let $z-f(y) = \varepsilon\,\delta(y-x)$, $w-f'(y) = \varepsilon\,\delta'(y-x)$. Then,

$$g(y, f(y)+\varepsilon\,\delta(y-x), f'(y)+\varepsilon\,\delta'(y-x)) \approx$$

$$g(y, f(y), f'(y)) + \frac{\partial g}{\partial f(y)} \cdot \varepsilon\,\delta(y-x) + \frac{\partial g}{\partial f'(y)}\,\varepsilon\,\delta'(y-x) \implies$$

$$\frac{\delta H}{\delta f(x)} \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \Big[g(y, f(y), f'(y)) + \frac{\partial g}{\partial f(y)} \cdot \varepsilon\,\delta(y-x)$$

$$+ \frac{\partial g}{\partial f'(y)}\,\varepsilon\,\delta'(y-x) - g(y, f(y), f'(y)) \Big] dy$$

$$\overset{\text{IBP}}{=} \int \left[\frac{\partial g}{\partial f(y)} \delta(y-x) - \frac{d}{dy} \frac{\partial g}{\partial f'(y)} \delta(y-x) \right] dy =$$

$$\boxed{\frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'} = \frac{\delta}{\delta f(x)} \int g(y, f(y), f'(y))\, dy}$$

Next, let $J[f] = \int g(y, f, f', f'')\, dy$. Then,

$$\frac{\delta J}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J[f(y) + \varepsilon \delta(y-x)] - J[f(y)] \right) =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \Big[g(y, f(y) + \varepsilon \delta(y-x), f'(y) + \varepsilon \delta'(y-x), f''(y) + \varepsilon \delta''(y-x)) -$$
$$g(y, f(y), f'(y), f''(y)) \Big] dy$$

We Taylor expand $g(y, a, b, c)$ near $a = f(y)$, $b = f'(y)$, $c = f''(y)$.

$$g(y, a, b, c) \approx g(y, f(y), f'(y), f''(y)) + \left(\frac{\partial g}{\partial a}\Big|_{(y, f(y), f'(y), f''(y))} \right)(a - f(y))$$

$$+ \left(\frac{\partial g}{\partial b}\Big|_{(y, f(y), f'(y), f''(y))} \right)(b - f'(y)) + \left(\frac{\partial g}{\partial c}\Big|_{(y, f(y), f'(y), f''(y))} \right)(c - f''(y)) =$$

Let $a - f(y) = \varepsilon \delta(y-x)$, $b - f'(y) = \varepsilon \delta'(y-x)$, $c - f''(y) = \varepsilon \delta''(y-x)$.

Then, we have

$$g(y, f(y)+\varepsilon\delta(y-x), f'(y)+\varepsilon\delta'(y-x), f''(y)+\varepsilon\delta''(y-x)) \approx$$

$$g(y, f(y), f'(y), f''(y)) + \frac{\partial g}{\partial f}\varepsilon\delta(y-x) + \frac{\partial g}{\partial f'}\varepsilon\delta'(y-x) + \frac{\partial g}{\partial f''}\varepsilon\delta''(y-x)$$

$$\Rightarrow \frac{\delta J[f]}{\delta f(x)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\Big[g(y, f(y), f'(y), f''(y)) +$$

$$\frac{\partial g}{\partial f}\varepsilon\delta(y-x) + \frac{\partial g}{\partial f'}\varepsilon\delta'(y-x) + \frac{\partial g}{\partial f''}\varepsilon\delta''(y-x) - g(y, f(y), f'(y), f''(y))\Big]dy$$

$$\overset{\text{IBP}}{=} \int\left[\frac{\partial g}{\partial f(y)}\delta(y-x) - \frac{d}{dy}\frac{\partial g}{\partial f'(y)}\delta(y-x) + \frac{d^2}{dy^2}\frac{\partial g}{\partial f''(y)}\delta(y-x)\right]dy =$$

$$\boxed{\frac{\partial g}{\partial f(x)} - \frac{d}{dx}\frac{\partial g}{\partial f'(x)} + \frac{d^2}{dx^2}\frac{\partial g}{\partial f''(x)} = \frac{\delta}{\delta f(x)}\int g(y, f(y), f'(y), f''(y))\,dy}$$

1.4

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \frac{\delta}{\delta \phi(y)} \int \phi(z)\, \delta(z-x)\, dz \overset{\text{Result 2}}{=} \delta(y-x).$$

$$\frac{\delta \dot{\phi}(t)}{\delta \phi(t_0)} = \frac{\delta}{\delta \phi(t_0)} \int \dot{\phi}(t')\, \delta(t'-t)\, dt' \overset{\text{Result 5}}{=} -\frac{d}{dt_0}\, \delta(t_0-t) =$$

$\dfrac{d}{dt}\, \delta(t_0-t)$, where the final equality can be understood with the chain rule.

1.5 See above.

1.61

Let $Z_0[J] = \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta(x-y)J(y)\right)$, where

$\Delta(x) = \Delta(-x)$. Then,

$$\frac{\delta Z_0[J]}{\delta J(z_1)} \overset{\text{chain rule}}{=} Z_0[J]\cdot\left(-\frac{1}{2}\right)\frac{\delta}{\delta J(z_1)}\int d^4x\, d^4y\, J(x)\Delta(x-y)J(y)$$

$$= -\frac{1}{2} Z_0[J] \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int d^4x\, d^4y \Big[(J(x)+\varepsilon\delta(x-z_1))\Delta(x-y)J(y)$$

$$- J(x)\Delta(x-y)J(y) + J(x)\Delta(x-y)(J(y)+\varepsilon\delta(y-z_1))$$

$$- J(x)\Delta(x-y)J(y)\Big] =$$

$$-\frac{1}{2} Z_0[J] \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int d^4x\, d^4y \Big[\varepsilon\delta(x-z_1)\Delta(x-y)J(y)$$

$$+ \varepsilon J(x)\Delta(x-y)\delta(y-z_1)\Big] =$$

$$-\frac{1}{2} Z_0[J]\left(\int d^4y\,[\Delta(z_1-y)J(y)] + \int d^4x\,[J(x)\Delta(x-z_1)]\right)$$

$$= -\frac{1}{2} Z_0[J]\left(\int d^4y\,[\Delta(z_1-y)J(y)] + \int d^4y\,[J(y)\Delta(y-z_1)]\right)$$

$$= \boxed{-Z_0[J]\int d^4y\,\Delta(z_1-y)J(y)}$$