Free-Space Diffusion Equation in 1D with δ-Function Initial Condition

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We consider the diffusion equation in one dimension

$$u_t(x,t) = Du_{xx}(x,t),$$

where $x \in (-\infty, +\infty), D > 0$, and $t > 0$. We solve this equation given the initial condition $u(x,0) = \delta(x)$.

To accomplish this goal, we shall use the Fourier transform and its inverse,

$$\mathcal{F}[f(x)] = \tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

$$\mathcal{F}^{-1}[\tilde{f}(k)] = f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}.$$

Crucially, it follows that $f''(x) = \mathcal{F}^{-1}[-k^2 \tilde{f}(k)]$.

Taking the Fourier transform of the diffusion equation, we have

$$\mathcal{F}[u_t(x,t)] = \mathcal{F}[Du_{xx}(x,t)] \iff \tilde{u}_t(k,t) = -Dk^2 \tilde{u}(k,t) \implies \tilde{u}(k,t) = \tilde{u}(k,0) e^{-Dk^2 t},$$

where

$$\tilde{u}(k,0) = \mathcal{F}[u(x,0)] = \int_{-\infty}^{\infty} dx \delta(x) e^{-ikx} = 1.$$

Thus, we have $\tilde{u}(k,t) = e^{-Dk^2 t}$, and our solution can be found by performing the inverse Fourier transform.

$$u(x,t) = \mathcal{F}^{-1}[\tilde{u}(k,t)] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-Dk^2 t} e^{ikx} = \boxed{\frac{1}{\sqrt{2\pi(2Dt)}} e^{-\frac{x^2}{2(2Dt)}}.}$$

This is a gaussian centered on the origin with standard deviation $\sigma(t) = \sqrt{2Dt}$.