<u>Exercise</u> We compute the electric field and potential of a uniformly charged sphere located at the origin.

Gauss's Law states $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \iff \int \vec{E} \cdot d\vec{s} = Q_{enc}/\varepsilon_0$.

For spherically symmetric problems, this simplifies to

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \hat{r}$$

Consider a uniformly charged sphere of charge Q, radius R located at the origin. Outside the sphere, the formula above implies

$$\vec{E}_{out} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r} .$$

Inside, the enclosed charge will be different. The sphere has charge density

$$\rho = Q/V = \frac{3Q}{4\pi R^3}$$

The enclosed charge at radius r is

$$Q_{enc} = \int_V \rho = \frac{3Q}{4\pi R^3} \cdot 4\pi \int_0^r \tilde{r}^2 \, d\tilde{r} = \frac{Q r^3}{R^3} \implies$$

$$\vec{E}_{in} = \frac{1}{4\pi\varepsilon_0 r^2} \cdot \frac{Qr^3}{R^3}\hat{r} = \frac{Qr}{4\pi\varepsilon_0 R^3}\hat{r} \Longrightarrow$$

$$\boxed{\vec{E} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}, & r > R \\ \frac{Qr}{4\pi\varepsilon_0 R^3}, & r < R \end{cases}}$$

We now find the electrostatic potential everywhere in space:

$$\Phi(r) = -\int \vec{E} \cdot \vec{dl} = -\int_{\infty}^{r} \vec{E} \cdot \vec{dr}.$$

Suppose $r > R$. Then,

$$\Phi(r) = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = \frac{Q}{4\pi\varepsilon_0 r}.$$

Suppose $r < R$. Then,

$$\Phi(r) = -\int_{\infty}^{R} E_{out}\,dr - \int_{R}^{r} E_{in}\,dr = \frac{Q}{4\pi\varepsilon_0 R} - \int_{R}^{r} \frac{Q\tilde{r}}{4\pi\varepsilon_0 R^3}\,d\tilde{r}$$

$$= \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 R^3}\left(\frac{r^2}{2} - \frac{R^2}{2}\right) = \frac{Q}{4\pi\varepsilon_0 R}\left(-\frac{r^2}{2R^2} + \frac{3}{2}\right).$$

Thus, the potential everywhere in space is given by

$$\boxed{\Phi(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r}, & r > R \\ \dfrac{Q}{4\pi\varepsilon_0 R}\left(-\dfrac{r^2}{2R^2} + \dfrac{3}{2}\right), & r < R \end{cases}}$$