Exercise | Suppose we have a spherically symmetric charge distribution

$$\rho(r) = Ae^{-\alpha r}$$

We compute its electric field and potential.

Gauss's law states $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \iff \int \vec{E} \cdot d\vec{S} = Q_{enc}/\varepsilon_0 =$

$E \cdot 4\pi r^2$, due to spherical symmetry. The charge enclosed by a sphere of radius r is given by

$$Q_{enc} = \int_V \rho = A \cdot 4\pi \int_0^r \tilde{r}^2 e^{-\alpha \tilde{r}} \, d\tilde{r} =$$

$$A \cdot 4\pi \int_0^r \partial_{\alpha\alpha} e^{-\alpha r} \, dr = A \cdot 4\pi \, \partial_{\alpha\alpha} \int_0^r e^{-\alpha \tilde{r}} \, d\tilde{r} =$$

$$A \cdot 4\pi \, \partial_{\alpha\alpha} \frac{1}{\alpha} \int_0^{\alpha r} e^{-x} \, dx = A \cdot 4\pi \, \partial_{\alpha\alpha} \frac{1}{\alpha} (1 - e^{-\alpha r})$$

$$= A \cdot 4\pi \cdot \frac{\partial}{\partial \alpha} \left[\frac{-1}{\alpha^2} (1 - e^{-\alpha r}) + \frac{1}{\alpha} (r e^{-\alpha r}) \right] =$$

$$4\pi A \left[\frac{2}{\alpha^3} (1 - e^{-\alpha r}) - \frac{1}{\alpha^2} r e^{-\alpha r} - \frac{1}{\alpha^2} (r e^{-\alpha r}) \right.$$

$$-\frac{1}{\alpha} r^2 e^{-\alpha r}\Big] = 4\pi A e^{-\alpha r}\left[\frac{2}{\alpha^3}\left(e^{\alpha r}-1\right) - \frac{2r}{\alpha^2} - \frac{r^2}{\alpha}\right].$$

Thus, the electric field is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^2} 4\pi A e^{-\alpha r}\left[\frac{2}{\alpha^3}\left(e^{\alpha r}-1\right) - \frac{2r}{\alpha^2} - \frac{r^2}{\alpha}\right]\hat{r}$$

$$= \boxed{\frac{A e^{-\alpha r}}{\varepsilon_0 r^2}\left[\frac{2}{\alpha^3}\left(e^{\alpha r}-1\right) - \frac{2r}{\alpha^2} - \frac{r^2}{\alpha}\right]\hat{r}}$$

As a sanity check, Mathematica happily confirms Gauss's law:

$$\frac{1}{r^2}\,\mathrm{D}\left[r^2\,\frac{A}{\epsilon_0}\,\frac{e^{-\alpha r}}{r^2}\left(\frac{2}{\alpha^3}\left(e^{\alpha r}-1\right) - \frac{2r}{\alpha^2} - \frac{r^2}{\alpha}\right),\ r\right]\ \texttt{// FullSimplify}$$

$$\frac{A\, e^{-r\alpha}}{\epsilon_0}$$

(where we have used the formula for the divergence in spherical coordinates).

We now compute the potential:

$$\Phi(r) = -\int_{\infty}^{r} E(r)\,dr \overset{\text{Mathematica}}{=} \boxed{\frac{A}{\varepsilon_0 \alpha^3}\left(\frac{2}{r} - \frac{2e^{-\alpha r}}{r} - \alpha e^{-\alpha r}\right)}$$

We can also easily check the correctness of the solution in Mathematica:

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1/r^2 D[r^2 D[(A e^(-r α) (-2 + 2 e^(r α) - r α))/(r α^3 ϵ0), r], r] // FullSimplify
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$$\frac{1}{r^2} D\left[r^2 D\left[\frac{A e^{-r\alpha}(-2 + 2 e^{r\alpha} - r\alpha)}{r\alpha^3 \epsilon_0}, r\right], r\right] \text{ // FullSimplify}$$

$$-\frac{A e^{-r\alpha}}{\epsilon_0}$$

Where, again, we have used the known formula for the Laplacian in spherical coordinates.

Generally speaking, the potential takes the following form: