<u>Exercise 1</u> We show that the eigenvectors of a Hermitian operator form an orthonormal set. (Source: Susskind, Theoretical Minimum, Vol. 2)

Suppose L is a Hermitian operator and $|\lambda_1\rangle$, $|\lambda_2\rangle$ are two distinct eigenvectors with eigenvalues λ_1 and λ_2 respectively. Then

$$\langle\lambda_1|L|\lambda_2\rangle = \lambda_2\langle\lambda_1|\lambda_2\rangle$$

However, $\langle\lambda_1|L|\lambda_2\rangle = (\langle\lambda_2|L^\dagger|\lambda_1\rangle)^* =$

$$(\langle\lambda_2|L|\lambda_1\rangle)^* = \lambda_1^*\langle\lambda_2|\lambda_1\rangle = \lambda_1\langle\lambda_1|\lambda_2\rangle$$

since the eigenvalues of a Hermitian operator are always real.

$$\Big(\text{Proof: } \langle\lambda|L|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = (\langle\lambda|L^\dagger|\lambda\rangle)^* =$$

$$(\langle\lambda|L|\lambda\rangle)^* = \lambda^*\langle\lambda|\lambda\rangle \implies \lambda = \lambda^* \Big)$$

Thus, we have shown

$$(\lambda_1 - \lambda_2)\langle\lambda_1|\lambda_2\rangle = 0.$$

Case 1: $\lambda_1 \neq \lambda_2$. Then $\langle\lambda_1|\lambda_2\rangle = 0$.

Case 2: $\lambda_1 = \lambda_2 = \lambda$. We construct two orthonormal vectors in the subspace.

Let $|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle$. Then,

$$L|A\rangle = \alpha L|\lambda_1\rangle + \beta L|\lambda_2\rangle = \lambda(\alpha|\lambda_1\rangle + \beta|\lambda_2\rangle)$$

So $|A\rangle$ will be an eigenvector of L for any α and β. We will let $|\tilde{\lambda}_1\rangle = \frac{|\lambda_1\rangle}{\sqrt{\langle\lambda_1|\lambda_1\rangle}}$ be our first orthonormal vector. Then, by Gram-Schmidt,

$$|\tilde{\lambda}_2\rangle = |\lambda_2\rangle - \mathrm{proj}_{|\tilde{\lambda}_1\rangle}|\lambda_2\rangle = |\lambda_2\rangle - |\tilde{\lambda}_1\rangle\langle\tilde{\lambda}_1|\lambda_2\rangle.$$

It follows that

$$\langle\tilde{\lambda}_1|\tilde{\lambda}_2\rangle = \langle\tilde{\lambda}_1|\lambda_2\rangle - \langle\tilde{\lambda}_1|\tilde{\lambda}_1\rangle\langle\tilde{\lambda}_1|\lambda_2\rangle$$

$$= 0.$$

Thus, we have shown that we can always make eigenvectors of Hermitian operators orthogonal. Making them orthonormal follows trivially. $\blacksquare$