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EGIAN: Chapter 1.4
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1a.) We show that if ϕ is a scalar, then so is $\vec{\nabla}\phi \cdot \vec{\nabla}\phi$.

We show this in two ways. First in vector form; second in component form.

We remark that since ϕ' is a scalar, $\phi' = \phi$.

Vector Form:

We are given $\vec{\nabla}' = R\vec{\nabla} \Rightarrow \vec{\nabla}'\phi' = R\vec{\nabla}\phi' = R\vec{\nabla}\phi$.

Thus,

$$\vec{\nabla}'\phi' \cdot \vec{\nabla}'\phi' = (R\vec{\nabla}\phi) \cdot (R\vec{\nabla}\phi) =$$

$$(R\vec{\nabla}\phi)^T(R\vec{\nabla}\phi) = \vec{\nabla}\phi^T R^T R\vec{\nabla}\phi = \vec{\nabla}\phi^T\vec{\nabla}\phi =$$

$$= \vec{\nabla}\phi \cdot \vec{\nabla}\phi.$$

Component Form:

$$\sum_{l} \left(\frac{\partial \phi'}{\partial x'^{l}}\right)^2 = \sum_{l,m} \delta^{lm} \frac{\partial \phi'}{\partial x'^{l}} \frac{\partial \phi'}{\partial x'^{m}} =$$

$$\sum_{l,m} \delta^{lm} R^{l\alpha} \frac{\partial \phi'}{\partial x^{\alpha}} R^{m\beta} \frac{\partial \phi'}{\partial x^{\beta}} = \underbrace{\sum_{l} R^{l\alpha} R^{l\beta}}_{(R^T R)^{\alpha\beta}} \frac{\partial \phi'}{\partial x^{\alpha}} \frac{\partial \phi'}{\partial x^{\beta}}$$

$$= \delta^{\alpha\beta} \frac{\partial \phi'}{\partial x^{\alpha}} \frac{\partial \phi'}{\partial x^{\beta}} =$$

$$\sum_{\alpha} \left(\frac{\partial \phi'}{\partial x^{\alpha}}\right)^2 = \sum_{\alpha} \left(\frac{\partial \phi}{\partial x^{\alpha}}\right)^2 . \qquad \blacksquare$$

Thus, we have shown that for a scalar ϕ, $\vec{\nabla}\phi \cdot \vec{\nabla}\phi$ is a scalar too.

1b.) We show that the Laplacian of a scalar is also a scalar.

Vector Form:

We know that $\vec{\nabla}' = R\vec{\nabla} \Rightarrow \nabla'^2 = \vec{\nabla}' \cdot \vec{\nabla}' =$

$R\vec{\nabla} \cdot R\vec{\nabla} = \vec{\nabla}^T R^T R \vec{\nabla} = \vec{\nabla}^T \vec{\nabla} = \vec{\nabla} \cdot \vec{\nabla} =$

∇^2. Thus, we have $\nabla'^2 \phi' = \nabla^2 \phi' = \nabla^2 \phi$. $\blacksquare$

Index Form:

$$\sum_l \frac{\partial^2 \phi'}{\partial (x'^l)^2} = \sum_{l,m} \delta^{lm} \frac{\partial^2 \phi'}{\partial x'^l \partial x'^m} =$$

$$\sum_{l,m} \delta^{lm} R^{l\alpha} \frac{\partial}{\partial x^\alpha} R^{m\beta} \frac{\partial}{\partial x^\beta} \phi' =$$

$$\sum_{l,m} \delta^{lm} R^{l\alpha} R^{m\beta} \frac{\partial^2 \phi'}{\partial x^\alpha \partial x^\beta} = \sum_l R^{l\alpha} R^{l\beta} \frac{\partial^2 \phi'}{\partial x^\alpha \partial x^\beta}$$

$$= \delta^{\alpha\beta} \frac{\partial^2 \phi'}{\partial x^\alpha \partial x^\beta} = \sum_\alpha \frac{\partial^2 \phi'}{\partial (x^\alpha)^2} = \sum_\alpha \frac{\partial^2 \phi}{\partial (x^\alpha)^2} .$$

Thus, we have also shown that the Laplacian of a scalar is a scalar.

2.

We show that S^{ij} is a tensor

$$S^{ij} = T^{ij} + T^{ji}$$

$$S'^{ij} = T'^{ij} + T'^{ji} = R^{i\alpha} R^{j\beta} T^{\alpha\beta} + R^{j\alpha} R^{i\beta} T^{\beta\alpha}$$

between these steps, we have used $\alpha \mapsto \beta$, $\beta \mapsto \alpha$ since they are both summed over

$$= R^{i\alpha} R^{j\beta} T^{\alpha\beta} + R^{j\beta} R^{i\alpha} T^{\beta\alpha} =$$

$$R^{i\alpha} R^{j\beta} \left(T^{\alpha\beta} + T^{\beta\alpha}\right) = R^{i\alpha} R^{j\beta} S^{\alpha\beta}. \quad \boxtimes$$

3.1

$$d^3x' = dx^{1\prime} dx^{2\prime} dx^{3\prime} = R^{1l} dx^l \, R^{2m} dx^m \, R^{3n} dx^n .$$

$dx^1 dx^2 = -dx^2 dx^1 \implies$ no terms like $dx^{1^2}, dx^{2^2}, dx^{3^2}$
$dx^{1^3}, dx^{2^3}, dx^{3^3}$

$l, m, n \in \{1, 2, 3\}$, so we expand this product, and simplify.

$$(R^{11} dx^1 + R^{12} dx^2 + R^{13} dx^3) \cdot$$
$$(R^{21} dx^1 + R^{22} dx^2 + R^{23} dx^3) \cdot$$
$$(R^{31} dx^1 + R^{32} dx^2 + R^{33} dx^3) =$$

$$\Big(\cancel{R^{11} dx^1 R^{21} dx^1}^{\,0} + R^{11} dx^1 R^{22} dx^2 + R^{11} dx^1 R^{23} dx^3 +$$
$$R^{12} dx^2 R^{21} dx^1 + \_\_ dx^{2^2} + R^{12} dx^2 R^{23} dx^3 +$$
$$R^{13} dx^3 R^{21} dx^1 + R^{13} dx^3 R^{22} dx^2 + \cancel{R^{13} dx^3 R^{23} dx^3}^{\,0} \Big)$$

$$(R^{31} dx^1 + R^{32} dx^2 + R^{33} dx^3) =$$

$$\Big[dx^1 dx^2 (R^{11}R^{22} - R^{12}R^{21}) + dx^1 dx^3 (R^{11}R^{23} - R^{13}R^{21})$$
$$+ dx^2 dx^3 (R^{12}R^{23} - R^{13}R^{22})\Big](R^{31}dx^1 + R^{32}dx^2 + R^{33}dx^3)$$
$$= \underline{\quad}\, {dx^1}^2 dx^2 + \underline{\quad}\, dx^1 {dx^2}^2 + dx^1 dx^2 dx^3\, R^{33}(R^{11}R^{22} - R^{12}R^{21})$$
$$+ \underline{\quad}\, {dx^1}^2 dx^3 + dx^1 dx^3 dx^2\, R^{32}(R^{11}R^{23} - R^{13}R^{21}) +$$
$$\underline{\quad}\, dx^1 {dx^3}^2 + dx^2 dx^3 dx^1\, R^{31}(R^{12}R^{23} - R^{13}R^{22}) + \underline{\quad} + \underline{\quad}$$
$$= dx^1 dx^2 dx^3 \Big[R^{33}(R^{11}R^{22} - R^{12}R^{21}) - R^{32}(R^{11}R^{23} - R^{13}R^{21})$$
$$+ R^{31}(R^{12}R^{23} - R^{13}R^{22})\Big] = dx^1 dx^2 dx^3 \det R = dx^1 dx^2 dx^3.$$

We know this is the determinant of R by using the recursive formula along the bottom row.

$$\begin{pmatrix} R^{11} & R^{12} & R^{13} \\ R^{21} & R^{22} & R^{23} \\ R^{31} & R^{32} & R^{33} \end{pmatrix}$$

$\blacksquare$

4.1

We show that the Laplace Runge-Lenz vector is conserved:

$$\vec{\mathcal{L}} = \vec{l} \times \dot{\vec{r}} + \frac{k}{r}\vec{r} = (\vec{r} \times \dot{\vec{r}}) \times \dot{\vec{r}} + \frac{k}{r}\vec{r}.$$

We will use the facts that $\ddot{\vec{r}} = -\frac{k\vec{r}}{r^3}$, $(\vec{A} \times \vec{B}) \times \vec{C} = \vec{A} \times (\vec{B} \times \vec{C}) - \vec{B} \times (\vec{A} \times \vec{C})$, and

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}).$$ It follows that

$$\dot{\vec{\mathcal{L}}} = (\cancelto{0}{\dot{\vec{r}} \times \dot{\vec{r}}} + \cancelto{0}{\vec{r} \times \ddot{\vec{r}}}) \times \dot{\vec{r}} + (\vec{r} \times \dot{\vec{r}}) \times \ddot{\vec{r}} + \frac{k}{r^2}(\dot{\vec{r}} r - \vec{r}\dot{r})$$

$$= (\vec{r} \times \dot{\vec{r}}) \times \ddot{\vec{r}} + \frac{k}{r^2}(\dot{\vec{r}} r - \vec{r}\dot{r}) =$$

$$\vec{r} \times (\dot{\vec{r}} \times \ddot{\vec{r}}) - \dot{\vec{r}} \times \cancelto{0}{(\vec{r} \times \ddot{\vec{r}})} + \frac{k}{r^2}(\dot{\vec{r}} r - \vec{r}\dot{r}) =$$

$$\dot{\vec{r}}(\vec{r} \cdot \ddot{\vec{r}}) - \ddot{\vec{r}}(\vec{r} \cdot \dot{\vec{r}}) + \frac{k\dot{\vec{r}}}{r} - \frac{k\vec{r}\dot{r}r}{r^3} =$$

$$-\cancel{\frac{k\dot{\vec{r}}r^2}{r^3}} + \frac{k\vec{r}}{r^3}(\vec{r} \cdot \dot{\vec{r}}) + \cancel{\frac{k\dot{\vec{r}}}{r}} - \frac{k\vec{r}\dot{r}r}{r^3} =$$

$$\frac{k\vec{r}}{r^3}(\vec{r}\cdot\dot{\vec{r}}) - \frac{k\vec{r}}{r^3}\dot{r}r = \frac{k\vec{r}}{r^3}(\vec{r}\cdot\dot{\vec{r}} - r\dot{r}) = 0$$

Since $\vec{r}^2 = r^2 \Rightarrow 2\vec{r}\cdot\dot{\vec{r}} = 2r\dot{r}$. $\square$

5.) $S^{ij} = T^{ij} + T^{ji}$. $A^{ij} = T^{ij} - T^{ji}$.

We wish to show that $S^{ij} A^{ij} = 0$. We recall $S^{ij} = S^{ji}$, $A^{ij} = -A^{ji}$.

$$S^{ij} A^{ij} = -S^{ij} A^{ji} = -S^{ji} A^{ji} = -S^{ij} A^{ij},$$

where for the final step, we used the symmetry of the treatment to relabel $i \leftrightarrow j$. Thus, $S^{ij} A^{ij} = -S^{ij} A^{ij} \Longleftrightarrow S^{ij} A^{ij} = 0$. $\boxtimes$

6. Let T^{ijk} be totally antisymmetric. Then T has $\frac{1}{3!}D(D-1)(D-2)$ components.

<u>Proof:</u> Since T^{ijk} is totally antisymmetric, no two indices may be equal and have T not vanish. So when we choose i, we have D choices. But this leaves only $D-1$ choices for j, and $(D-1)-1=D-2$ choices for k. Finally, once we have T^{ijk}, we also have T^{jik}, T^{ikj}, ... for all permutations of ijk by antisymmetry. There are $3!$ permutations of 3 elements, so T^{ijk} has $\frac{1}{3!}D(D-1)(D-2)$ components. $\square$

For $D=3$, T^{123} is that component.

7. We show that the only element of T^{ijk} transforms as a scalar under rotation.

$$T'^{123} = R^{1\alpha}R^{2\beta}R^{3\gamma}T^{\alpha\beta\gamma}$$

$$= R^{11}R^{22}R^{33}T^{123} + R^{11}R^{23}R^{32}T^{132} + R^{12}R^{21}R^{33}T^{213} + R^{12}R^{23}R^{31}T^{231} + R^{13}R^{21}R^{32}T^{312} + R^{13}R^{22}R^{31}T^{321} =$$

$$T^{123}\left(R^{11}R^{22}R^{33} + R^{12}R^{23}R^{31} + R^{13}R^{21}R^{32} - R^{11}R^{23}R^{32} - R^{12}R^{21}R^{33} - R^{13}R^{22}R^{31}\right) =$$

$$T^{123}\left[R^{11}\left(R^{22}R^{33} - R^{23}R^{32}\right) + R^{12}\left(R^{23}R^{31} - R^{21}R^{33}\right) + R^{13}\left(R^{21}R^{32} - R^{22}R^{31}\right)\right] = T^{123}\det R = T^{123}$$

8.) Let $H^{k\cdot ij} = G^{ki\cdot j} + G^{kj\cdot i}$. We show that

$$G^{ki\cdot j} = \frac{1}{2}\left(H^{k\cdot ij} + H^{i\cdot jk} - H^{j\cdot ki}\right).$$ (We will use the fact that $A^{ij\cdot k} = A^{ji\cdot k}$.)

Proof:

$$\frac{1}{2}\left(H^{k\cdot ij} + H^{i\cdot jk} - H^{j\cdot ki}\right) = \frac{1}{2}\left(G^{ki\cdot j} + \cancel{G^{kj\cdot i}} + \cancel{G^{ij\cdot k}} + G^{ik\cdot j} - \cancel{G^{jk\cdot i}} - \cancel{G^{ji\cdot k}}\right) =$$

$$\frac{1}{2}\left(G^{ki\cdot j} + G^{ik\cdot j}\right) = G^{ki\cdot j} \qquad \blacksquare$$

9.) We show $\varepsilon^{ij} R^{ip} R^{jq} = \varepsilon^{pq}$.

$$\varepsilon^{ij} R^{ip} R^{jq} = \varepsilon^{11}\_ + \varepsilon^{12} R^{1p} R^{2q} + \varepsilon^{21} R^{2p} R^{1q} + \varepsilon^{22}\_\_$$

$$= R^{1p} R^{2q} - R^{2p} R^{1q}.$$

$$R = \begin{pmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{pmatrix} \Rightarrow \det R = R^{11} R^{22} - R^{12} R^{21}$$

If $p=1, q=2$, then $R^{1p} R^{2q} - R^{2p} R^{1q} = R^{11} R^{22} - R^{21} R^{12} = \det R$.

Let $p=2, q=1$, then $R^{1p} R^{2q} - R^{2p} R^{1q} = R^{12} R^{21} - R^{22} R^{11} = -\det R$.

Let $p=q=1$. Then $R^{1p} R^{2q} - R^{2p} R^{1q} = R^{11} R^{21} - R^{21} R^{11} = 0$.

Let $p=q=2$. Then $R^{1p} R^{2q} - R^{2p} R^{1q} = R^{12} R^{22} - R^{22} R^{12} = 0$.

Thus, $R^{1p} R^{2q} - R^{2p} R^{1q} = \varepsilon^{pq} \det R = \varepsilon^{pq}$ ☒

We next verify $\varepsilon^{ijk} R^{ip} R^{jq} R^{kr} = \varepsilon^{pqr}$.

$$\varepsilon^{ijk} R^{ip} R^{jq} R^{kr} = \varepsilon^{111}\_ + \varepsilon^{112}\_ + \varepsilon^{113}\_ + \varepsilon^{121}\_ +$$

$$\varepsilon^{122}\_ + \varepsilon^{123} R^{1p} R^{2q} R^{3r} + \varepsilon^{131}\_ + \varepsilon^{132} R^{1p} R^{3q} R^{2r}$$

$$+\varepsilon^{133}\_ + \varepsilon^{211}\_ + \varepsilon^{212}\_ + \varepsilon^{213} R^{2p} R^{1q} R^{3r} +$$

$$\varepsilon^{221}\_ + \varepsilon^{222}\_ + \varepsilon^{223}\_ + \varepsilon^{231} R^{2p} R^{3q} R^{1r} +$$

$$\varepsilon^{232}\_ + \varepsilon^{233}\_ + \varepsilon^{311}\_ + \varepsilon^{312} R^{3p} R^{1q} R^{2r} +$$

$$\varepsilon^{313}\_ + \varepsilon^{321} R^{3p} R^{2q} R^{1r} + \varepsilon^{322}\_ + \varepsilon^{323}\_ +$$

$$\varepsilon^{331}\_ + \varepsilon^{332}\_ + \varepsilon^{333} =$$

$$\varepsilon^{123} R^{1p} R^{2q} R^{3r} + \varepsilon^{132} R^{1p} R^{3q} R^{2r} +$$

$$\varepsilon^{213} R^{2p} R^{1q} R^{3r} + \varepsilon^{231} R^{2p} R^{3q} R^{1r} +$$

$$\varepsilon^{312} R^{3p} R^{1q} R^{2r} + \varepsilon^{321} R^{3p} R^{2q} R^{1r} =$$

$$R^{1p} R^{2q} R^{3r} + R^{2p} R^{3q} R^{1r} + R^{3p} R^{1q} R^{2r} -$$

$$R^{1p} R^{3q} R^{2r} - R^{2p} R^{1q} R^{3r} - R^{3p} R^{2q} R^{1r} =$$

$$R^{1p}\left(R^{2q} R^{3r} - R^{3q} R^{2r}\right) +$$

$$R^{2p}\left(R^{3q} R^{1r} - R^{1q} R^{3r}\right) +$$

$R^{3p}(R^{1q}R^{2r} - R^{2q}R^{1r})$. We can verify in MMA that for positive permutations of $pqr \in \{1,2,3\}$, this equals $\det R$, for negative permutations, it equals $-\det R$, and when there are repeated indices, it vanishes.

Thus, we conclude

$$
\begin{aligned}
&R^{1p}(R^{2q}R^{3r} - R^{3q}R^{2r}) + \\
&R^{2p}(R^{3q}R^{1r} - R^{1q}R^{3r}) + \\
&R^{3p}(R^{1q}R^{2r} - R^{2q}R^{1r}) = \varepsilon^{pqr}\det R = \varepsilon^{pqr} = \\
&\varepsilon^{ijk}R^{ip}R^{jq}R^{kr}
\end{aligned}
$$

$\square$