1.) We show that $\begin{pmatrix} p^2 q^3 \\ p^3 q^1 \\ p^1 q^2 \end{pmatrix}$ is not a vector. A vector is an object which transforms under rotation such that each component becomes a linear combination of other components

$$(x^i)' = R^{ij} x^j .$$

We look at the first component under transformation:

$$(p^2)'(q^3)' = R^{2j}p^j R^{3k}q^k =$$

$$R^{21}R^{31}p^1q^1 + R^{21}R^{32}p^1q^2 + R^{21}R^{33}p^1q^3 +$$

$$R^{22}R^{31}p^2q^1 + R^{22}R^{32}p^2q^2 + R^{22}R^{33}p^2q^3 +$$

$$R^{23}R^{31}p^3q^1 + R^{23}R^{32}p^3q^2 + R^{23}R^{33}p^3q^3$$

This sum contains terms, like $p^i q^i$, which are not components of the original array, and hence the array cannot be a vector.

By contrast, $p^2q^3 - p^3q^2 \mapsto A(p^2q^3 - p^3q^2) + B(p^3q^1 - p^1q^3)$
$+ C(p^1q^2 - p^2q^1)$, where A, B, and C

can be written in terms of the components of R. So it is a vector. $\boxed{2}$

2.1

We show that $\delta(x)\,\delta(y)$ is invariant under rotation around the origin. We consider the rotate coordinates

$$x' = \cos\theta\, x + \sin\theta y, \quad y' = -\sin\theta x + \cos\theta y.$$

We know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x')\,\delta(y')\,dx'\,dy' = 1$$

We now express this integral in terms of the un-primed variables.

$$dx'\,dy' = d(x\cos\theta + y\sin\theta)\,d(-\sin\theta x + \cos\theta y) =$$

$$(\cos\theta\,dx + \sin\theta\,dy)(-\sin\theta\,dx + \cos\theta\,dy) =$$

$$-\sin\theta\cos\theta\,dx^2 + \cos^2\theta\,dx\,dy - \sin^2\theta\,dy\,dx + \sin\theta\cos\theta\,dy^2$$

$$= (\cos^2\theta + \sin^2\theta)\,dx\,dy = dx\,dy,$$

where we have used $dxdy = -dydx$ and thrown away terms like dx^2, dy^2. Thus,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x')\,\delta(y')\,dx'dy' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x')\,\delta(y')\,dx\,dy =$$

$$= \iint \delta(\cos\theta\, x + \sin\theta\, y)\,\delta(-\sin\theta\, x + \cos\theta\, y)\,dx\,dy =$$

$$\iint \delta(\cos\theta\,(x + \tan\theta\, y))\,\delta(-\sin\theta\, x + \cos\theta\, y)\,dx\,dy =$$

$$\frac{1}{|\cos\theta|}\iint \delta(x + \tan\theta\, y)\,\delta(-\sin\theta\, x + \cos\theta\, y)\,dx\,dy =$$

$$\frac{1}{|\cos\theta|}\int dy\,\delta(-\sin\theta(-\tan\theta\, y) + \cos\theta\, y) =$$

$$\frac{1}{|\cos\theta|}\int dy\,\delta\left(\frac{y}{\cos\theta}(\sin^2\theta + \cos^2\theta)\right) =$$

$$\frac{1}{|\cos\theta|}\int dy\,\delta\left(y \cdot \frac{1}{\cos\theta}\right) = \frac{|\cos\theta|}{|\cos\theta|}\int dy\,\delta(y) = 1$$

$$= \iint dx\,dy\; \delta(x)\,\delta(y).$$

Thus, we have shown $\delta(x)\,\delta(y) = \delta(x')\,\delta(y')$

for any rotation about the origin.

3.1

Find matrix for rotation about x-axis by θ_x.

$$R_x(\theta_x) = e^{\theta_x J_x}, \quad J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We can simply find this in Mathematica:

$$R_x(\theta_x) = \text{MatrixExp}[\theta_x J_x] =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

$\boxtimes$

Similarly

$$R_y(\theta_y) = \exp \theta_y J_y = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$R_x(\theta_x) R_y(\theta_y) - R_y(\theta_y) R_x(\theta_x) =$$

$$\begin{pmatrix} 0 & -\sin\theta_x \sin\theta_y & -\sin\theta_y + \cos\theta_x \sin\theta_y \\ \sin\theta_x \sin\theta_y & 0 & -\sin\theta_x + \cos\theta_y \sin\theta_x \\ -\sin\theta_y + \cos\theta_x \sin\theta_y & -\sin\theta_x + \cos\theta_y \sin\theta_x & 0 \end{pmatrix}$$

$\neq 0$. Thus, we have shown that $[R_x(\theta_x), R_y(\theta_y)] \neq 0$.

$\boxed{}$

4.) Like a coward, I opted to skip this problem. I will return to it.

5.)

We show that $\langle \vec{p}^i \vec{p}^j \rangle$ over the direction of p is given by $\frac{1}{4\pi}\int \sin\theta \, d\theta \, d\phi \; \vec{p}^i \vec{p}^j = \frac{1}{3} \vec{p}^2 \delta^{ij}$.

We do this exhaustively.

```
In[4]:= Table[1/(4 π) ∫_0^(2 π) ∫_0^π (pi) (pj) Sin[θ] dθ dϕ,
    {pi, {p Sin[θ] Cos[ϕ], p Sin[θ] Sin[ϕ], p Cos[θ]}},
    {pj, {p Sin[θ] Cos[ϕ], p Sin[θ] Sin[ϕ], p Cos[θ]}}] //
   MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \frac{p^2}{3} & 0 & 0 \\ 0 & \frac{p^2}{3} & 0 \\ 0 & 0 & \frac{p^2}{3} \end{pmatrix}$$

$$= \frac{1}{3} \vec{p}^2 \delta^{ij}$$