1.)

a.) Show $\int dx\, \delta(x) f(x) = f(0)$

$$\delta(x) = \lim_{\tau \to 0} \begin{cases} 1/2\tau, & -\tau \leq x \leq \tau \\ 0, & \text{else} \end{cases}$$

$$\int_{-\tau}^{\tau} \delta(x) = \frac{1}{2\tau} \int_{-\tau}^{\tau} dx = 1.$$

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x) = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(x)\, dx \approx \lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(0)\, dx$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \cdot f(0) \cdot 2\tau = f(0)$$

$\boxtimes$

1b.] Show $\delta(ax) = \frac{1}{|a|}\delta(x)$ using

$$\int_{-\infty}^{\infty} dx\, \delta(ax) f(x) = \int_{-\infty}^{\infty} d(ax)\cdot \frac{1}{a}\delta(ax) f\left(\frac{ax}{a}\right) \equiv$$

$$\int_{-\infty}^{\infty} du \cdot \frac{1}{a}\delta(u) f(u/a) = \frac{1}{a}\lim_{\tau\to 0}\int_{-\tau}^{\tau}\frac{1}{2\tau} f(u/a) \approx$$

$$\frac{1}{a}\lim_{\tau\to 0}\frac{1}{2\tau} f(0)\cdot 2\tau = \frac{1}{a} f(0) = \frac{1}{a}\int_{-\infty}^{\infty} dx\, \delta(x) f(x)$$

$$\Rightarrow \frac{1}{a}\delta(x) = \delta(ax)$$

something in the chain rule requires a bit of subtlety, clearly.

2.

We determine $r(\theta)$ for an orbit. Let $u(\theta) = 1/r(\theta)$.

Our force equation has become $\ddot{r} = \dfrac{l^2}{r^3} - \dfrac{k}{r^2} = l^2u^3 - ku^2$.

We now express $\ddot{r}$ in terms of $u(\theta)$.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta} \cdot lu^2 \quad \text{in this problem.}$$

It follows that

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{d\theta}lu^2\right) =$$

$$l\left(\frac{d}{dt}\left(\frac{dr}{d\theta}\right)u^2 + \frac{dr}{d\theta}\frac{d}{dt}(u^2)\right) =$$

$$l\left(\frac{d^2r}{d\theta^2}\dot{\theta}u^2 + \frac{dr}{d\theta} \cdot 2u\dot{u}\right) =$$

$$l\left(\frac{d^2r}{d\theta^2} lu^4 + \frac{dr}{d\theta}\cdot 2u\frac{du}{d\theta}\dot{\theta}\right) =$$

$$l\left(r'' lu^4 + r'\cdot 2u^3 u' l\right).$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{1}{u^2}u'.$$

$$\frac{d^2r}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{1}{u^2}u'\right) = \frac{2}{u^3}(u')^2 - \frac{1}{u^2}u'' \Rightarrow$$

$$\ddot{r} = l\left[\left(\frac{2}{u^3}(u')^2 - \frac{1}{u^2}u''\right) lu^4 - \frac{1}{\cancel{u^2}}u'\cdot 2u^{\cancel{3}}u' l\right]$$

$$= l\left[2lu(u')^2 - u'' lu^2 - 2lu(u')^2\right] =$$

$-l^2u^2u''$. Finally, we plug back in to our force equation:

$$-l^2 \cancel{u^2} u'' = l^2 u^{\cancel{3}} - k \cancel{u^2} \implies$$

$$u'' = -u + k/l^2 \implies u(\theta) = \frac{k}{l^2} + C\cos(\theta + \delta$$

$$\implies \boxed{r(\theta) = \frac{1}{\frac{k}{l^2} + C\cos(\theta + \delta)}}$$

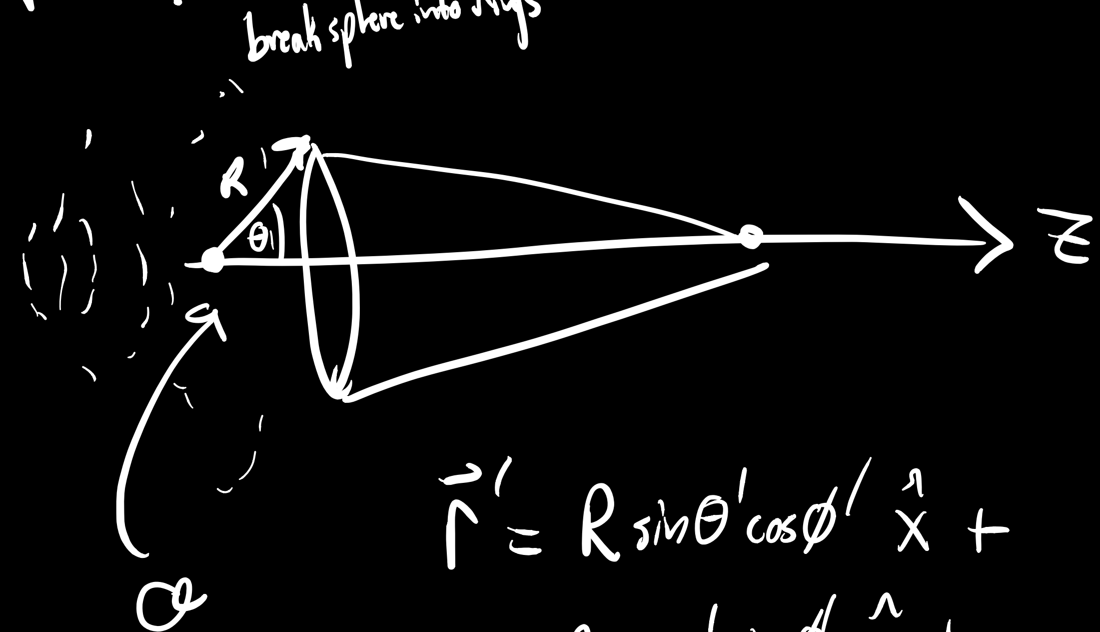
Where C and δ are determined by initial conditions

3. I was unable to solve this, but I hope to return.

4.] Let ρ be the uniform mass density of a sphere of rad R.

We consider the point $\vec{r} = z\hat{z}$, $z > R$.

$\vec{r}\,' = ?$



$$\vec{r}\,' = R\sin\theta'\cos\phi'\,\hat{x} + R\sin\theta'\sin\phi'\,\hat{y} + R\cos\theta'\,\hat{z}$$

$$\vec{r} - \vec{r}\,' = -R\sin\theta'\cos\phi'\,\hat{x} - R\sin\theta'\sin\phi'\,\hat{y} + (z - R\cos\theta)\hat{z}$$

$$\frac{\vec{F}}{-mG} = \int dv' \rho \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \implies$$

$$\frac{\vec{F}}{-mG\rho} = \int dv' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$|\vec{r}-\vec{r}'|^2 = R^2 \sin^2\theta' \cos^2\phi' + R^2 \sin^2\theta' \sin^2\phi' +$$

$$(z - R\cos\theta')^2 = R^2 \sin^2\theta' +$$

$$z^2 - 2zR\cos\theta' + R^2\cos^2\theta' = z^2 - 2zR\cos\theta + R^2$$

$$\implies \int dv' \frac{\vec{r}-\vec{r}'}{(z^2 - 2zR\cos\theta + R^2)^{3/2}} =$$

integrate over volume of sphere

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r'^2 \sin\theta' dr' d\theta' d\phi' \cdot \frac{1}{(z^2 - 2zR\cos\theta + R^2)^{3/2}}$$

$$\left(-R\sin\theta' \cancel{\cos\phi'}^{\,0} \hat{x} - R\sin\theta' \cancel{\sin\phi}^{\,0} \hat{y} + (z - R\cos\theta')\hat{z} \right)$$

$$= \frac{2\pi R^3}{3} \int_0^{\pi} \sin\theta\,(z - R\cos\theta) \cdot \frac{1}{(z^2 - 2zR\cos\theta + R^2)^{3/2}}\, d\theta\, \hat{z}$$

$$= \frac{2}{z^2} \cdot \frac{2\pi R^3}{3} \hat{z} = \frac{4\pi}{3} R^3 \cdot \frac{1}{z^2} \hat{z} \Rightarrow$$

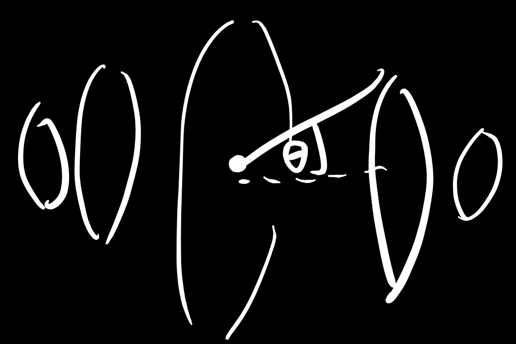
$$\vec{F} = -mG\rho V \cdot \frac{1}{z^2} \hat{z} = -\frac{GMm}{z^2} \hat{z}$$

This is the formula for the gravitational force due to a point charge located at the origin. $\boxtimes$

5. There is no grav force inside a spherical shell.
Rad R. Surface mass-density σ.

Let $\vec{r} = z\hat{z}$, $z<R$.

$\vec{r}' = \_\_\ \hat{x} + \_\_\ \hat{y} + (z - R\cos\theta)\hat{z}$



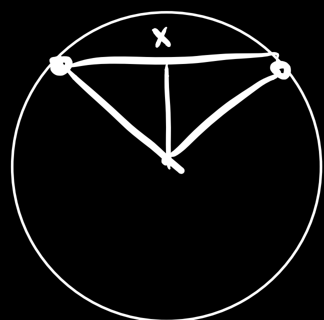
$$\frac{\vec{F}}{-Gm\rho} = \frac{2\pi R^3}{3}\hat{z}\int_0^{\pi} \frac{\sin\theta(z-R\cos\theta)}{(z^2-2zR\cos\theta+R^2)^{3/2}}\,d\theta$$

Same integral, diff and assumption. Should vanish

$$\frac{-1+\text{Sign}(R+z>0)}{z^2} = \frac{-1+1}{z^2} = 0 \quad \blacksquare$$

6.

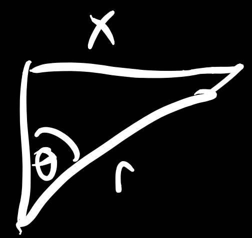
Engineers build a straight tunnel through Earth. We determine how long the trip takes between two points. WLOG, we pick these points to be at constant $\theta = \theta_0$.



By Newton's superb theorems, the particle only feels a gravitational force from the mass radially inward, as if it is concentrated at the center of the sphere.

$$\vec{F} = -\frac{G\rho \cdot \frac{4}{3}\pi r(x)^3}{r(x)^2}\hat{r} = -\frac{4}{3}\pi G\rho\, r(x)\,\hat{r}.$$

We find $r(x)$:



$r\sin\theta = x \Rightarrow r(x) = \dfrac{x}{\sin\theta}$ ..

We are interested in the force along the tunnel. $\hat{r} = \sin\theta\,\hat{x} + \ldots$

So, along the tunnel, we have

$$m\ddot{x} = \frac{-4}{3}\pi G\rho x \cdot m \Longrightarrow$$

$$\ddot{x} = \frac{-4}{3}\pi G\rho x$$

It follows that x will be periodic with angular frequency

$$\omega = \sqrt{\frac{4}{3}\pi G\rho} \quad \text{and hence half-period of} \quad \frac{\pi}{\sqrt{\frac{4}{3}\pi G\rho}}$$

$$\boxed{= \pi\sqrt{\frac{R}{g}} \sim 42 \text{ minutes, } 10 \text{ seconds}}$$

Interestingly, doesn't depend on choice of cities!