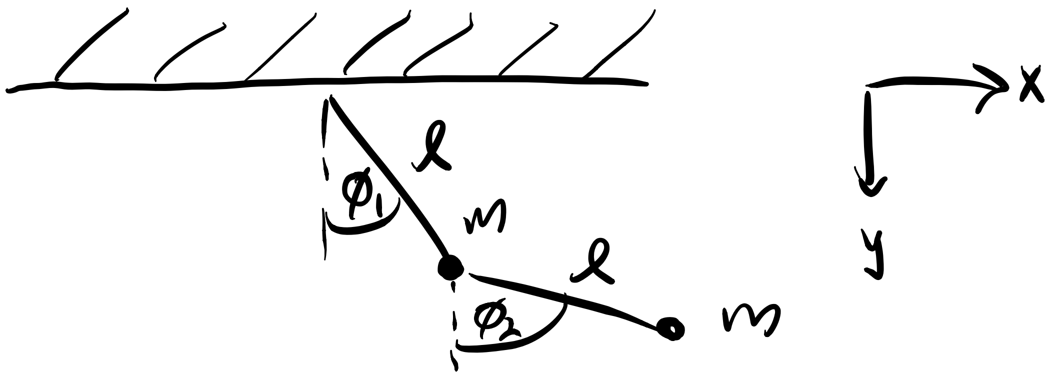
4.

We find the Lagrangian for the double pendulum parameterized as follows:



We first find the kinetic energy $T = \frac{1}{2} m(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2})$, where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the upper and lower blobs respectively.

$$\vec{r}_1 = l\sin\phi_1\,\hat{x} + l\cos\phi_1\,\hat{y}$$

$$\vec{r}_2 = \vec{r}_1 + l\sin\phi_2\,\hat{x} + l\cos\phi_2\,\hat{y} =$$

$$l(\sin\phi_1 + \sin\phi_2)\hat{x} + l(\cos\phi_1 + \cos\phi_2)\hat{y}.$$

It follows that

$$\dot{\vec{r}}_1 = l\cos\phi_1\,\dot{\phi}_1\,\hat{x} - l\sin\phi_1\,\dot{\phi}_1\,\hat{y} \implies$$

$$\dot{\vec{r}}_1^{\,2} = l^2(\cos^2\phi_1\,\dot{\phi}_1^2 + \sin^2\phi_1\,\dot{\phi}_1^2) = l^2\dot{\phi}_1^2$$

$$\dot{\vec{r}}_2 = l(\cos\phi_1\dot{\phi}_1 + \cos\phi_2\dot{\phi}_2)\hat{x} - l(\sin\phi_1\dot{\phi}_1 + \sin\phi_2\dot{\phi}_2)\hat{y}$$

$$\Rightarrow \dot{r}_2^{\,2} = l^2(\cos^2\phi_1\dot{\phi}_1^2 + 2\cos\phi_1\cos\phi_2\dot{\phi}_1\dot{\phi}_2 + \cos^2\phi_2\dot{\phi}_2^2 +$$

$$\sin^2\phi_1\dot{\phi}_1^2 + 2\sin\phi_1\sin\phi_2\dot{\phi}_1\dot{\phi}_2 + \sin^2\phi_2\dot{\phi}_2^2) =$$

$$l^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2).$$

So, we have

$$T = \frac{1}{2}ml^2(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2).$$

Next, we find the potential energies, which are determined by the vertical coordinate:

$$U = -mgy_1 - mgy_2 = -mgl\cos\phi_1 -$$

$$mgl(\cos\phi_1 + \cos\phi_2) = -2mgl\cos\phi_1 - mgl\cos\phi_2.$$

Thus, our Lagrangian is given by

$$\boxed{\begin{gathered}L = \frac{1}{2}ml^2(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2)\\ + 2mgl\cos\phi_1 + mgl\cos\phi_2\end{gathered}}$$

We now find the equations of motion.

$$\frac{\partial L}{\partial \phi_1} = \frac{1}{\cancel{2}} m l^2 \left(\cdot - \cancel{2} \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right) - 2 m g l \sin \phi_1$$

$$= m l^2 \left(- \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 - 2 \frac{g}{l} \sin \phi_1 \right).$$

$$\frac{\partial L}{\partial \dot{\phi}_1} = \frac{1}{\cancel{2}} m l^2 \left(\overset{2}{\cancel{4}} \dot{\phi}_1 + \cancel{2} \cos(\phi_1 - \phi_2) \dot{\phi}_2 \right) \Longrightarrow$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_1} = m l^2 \Big(2 \ddot{\phi}_1 - \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2) \dot{\phi}_2 + \cos(\phi_1 - \phi_2) \ddot{\phi}_2 \Big).$$

Thus, the Lagrange equation for ϕ_1 is given by

$$\boxed{\begin{gathered} - \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 - 2 \frac{g}{l} \sin \phi_1 = \\ 2 \ddot{\phi}_1 - \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2) \dot{\phi}_2 + \cos(\phi_1 - \phi_2) \ddot{\phi}_2 \end{gathered}}$$

$$\frac{\partial L}{\partial \phi_2} = \frac{1}{\cancel{2}} m l^2 \cdot \cancel{2} \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 - m g l \sin \phi_2 .$$

$$= m l^2 \left(\sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 - \frac{g}{l} \sin \phi_2 \right)$$

$$\frac{\partial L}{\partial \dot{\phi}_2} = \frac{1}{\cancel{2}} m l^2 \left(\cancel{2}\dot{\phi}_2 + \cancel{2}\cos(\phi_1 - \phi_2)\dot{\phi}_1 \right) =$$

$$m l^2 \left(\dot{\phi}_2 + \cos(\phi_1 - \phi_2)\dot{\phi}_1 \right) \implies$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} = m l^2 \Big[\ddot{\phi}_2 - \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\dot{\phi}_1 + \cos(\phi_1 - \phi_2)\ddot{\phi}_1 \Big]$$

Thus, the Lagrange equation for ϕ_2 is given by

$$\boxed{\sin(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 - \frac{g}{l}\sin\phi_2 = \ddot{\phi}_2 - \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\dot{\phi}_1 + \cos(\phi_1 - \phi_2)\ddot{\phi}_1}$$