Diffusion in a Very Viscous Fluid

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(Source: David Tong, Lectures on Kinetic Theory)

Exercise:

We analyze the Langevin equation in the case of a very viscous fluid.

Solution:

A particle in a noisy environment obeys the Langevin equation

$$m\ddot{\vec{x}} = -\gamma\dot{\vec{x}} - \vec{\nabla}V + \vec{f}(t)$$

where V is an external potential, and $\vec{f}(t)$ is a stochastic force modeling the noisy environment.

We make several simplifying assumptions. Firstly, there is no external force, so $V = 0$. Secondly, the fluid is assumed to be very viscous meaning that the drag dominates the inertial term. In practice, this amounts to setting $m = 0$. The Langevin equation thus reduces to

$$\gamma\dot{\vec{x}} = \vec{f}(t) \implies \vec{x}(t) = \vec{x}(0) + \frac{1}{\gamma}\int_0^t \vec{f}(t')dt'.$$

To solve the Langevin equation, we must make assumptions about the stochastic term. We assume the limit of white noise, meaning it has mean zero and is uncorrelated in space and time,

$$\langle \vec{f}(t)\rangle = 0 \quad , \quad \langle \vec{f}(t_1)\vec{f}(t_2)\rangle = 2\gamma^2 D\delta_{ij}\delta(t_2 - t_1).$$

(We are only examining the motion of the particle over time-scales much longer than the duration of a single collision.)

We now compute average values of the particle's trajectory. Firstly, the average position is given by

$$\langle \vec{x}(t)\rangle = \left\langle \vec{x}(0) + \frac{1}{\gamma}\int_0^t \vec{f}(t')dt' \right\rangle = \langle \vec{x}(0)\rangle + \int_0^t \langle \vec{f}(t')\rangle dt' \implies$$
$$\boxed{\langle \vec{x}(t)\rangle = \vec{x}(0).}$$

The variance of the trajectory is given by

$$\langle(\vec{x}(t)-\vec{x}(0))^2\rangle = \langle\vec{x}(t)^2\rangle - \vec{x}(0)^2.$$

Focusing on the first term, we have

$$\langle\vec{x}(t)^2\rangle = \left\langle\left(\vec{x}(0)+\frac{1}{\gamma}\int_0^t \vec{f}(t_1')dt'\right)\left(\vec{x}(0)+\frac{1}{\gamma}\int_0^t \vec{f}(t_2')dt'\right)\right\rangle =$$
$$\vec{x}(0)^2 + \frac{2}{\gamma}\vec{x}(0)\cdot\int_0^t\langle\vec{f}(t')\rangle dt' + \frac{1}{\gamma^2}\int_0^t\int_0^t dt_1'dt_2'\langle\vec{f}(t_1')\cdot\vec{f}(t_2')\rangle \implies$$
$$\langle(\vec{x}(t)-\vec{x}(0))^2\rangle = \frac{1}{\gamma^2}\int_0^t\int_0^t dt_1'dt_2'\sum_{i=1}^{3}\langle f_i(t_1')f_i(t_2')\rangle = \int_0^t dt\sum_{i=1}^{3}2D \implies$$
$$\boxed{\langle(\vec{x}(t)-\vec{x}(0))^2\rangle = 6Dt.}$$