We derive the density of states for a relativistic particle in a 3D box with periodic boundary conditions.

We recall that the relativistic energy is given by

$$E^2 = (\vec{p}c)^2 + (mc^2)^2 = (\hbar \vec{k} c)^2 + (mc^2)^2$$

The assumption of periodic boundary conditions (derived elsewhere) implies that $k_i = 2\pi n_i / L \implies \vec{k}^2 = 4\pi^2/L^2 \cdot \vec{n}^2$, so the energy is given by

$$E = \left[(2\pi\hbar c\vec{n}/L)^2 + (mc^2)^2\right]^{1/2}.$$

We recall that the density of states is the number of energy eigenstates between E and $E+dE$. Since $E = E(\vec{n}^2)$, and $n_i \in \mathbb{Z}$, this will correspond to a spherical shell in n-space, with volume $4\pi n(E)^2 dn(E)$.

$$n^2 = \left(E^2 - (mc^2)^2\right)\left(\frac{L}{2\pi\hbar c}\right)^2 \implies$$

$$dn = \frac{1}{2n}\left(\frac{L}{2\pi\hbar c}\right)^2 \cdot 2E\,dE =$$

$$\cancel{\frac{2\pi\hbar c}{L}}\left(E^2 - (mc^2)^2\right)^{-1/2}\left(\frac{L}{2\pi\hbar c}\right)^{\cancel{2}} E\,dE.$$

It follows that the density of states is given by

$$4\pi n^2 dn =$$

$$4\pi \left(E^2-(mc^2)^2\right)\left(\frac{L}{2\pi\hbar c}\right)^2 \left(E^2-(mc^2)^2\right)^{-1/2} \frac{L}{2\pi\hbar c} E\,dE$$

$$= \cancel{4\pi} \sqrt{E^2-(mc^2)^2}\ \frac{L^3}{2\cancel{8}\cancel{\pi^3}\pi^2\hbar^3c^3} E\,dE =$$

$$\boxed{\frac{V}{2\pi^2\hbar^3c^3}\sqrt{E^2-(mc^2)^2}\ E\,dE = g(E)\,dE}$$

It follows trivially that the density of states for massless relativistic particles is given by

$$\boxed{g(E)\,dE = \frac{VE^2}{2\pi^2\hbar^3c^3}\,dE}$$