Classical Atoms Are Unstable

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Exercise:

We compute the decay time for an atom according to classical electromagnetic theory.

Solution:

Classically, an accelerating charge will radiate energy

$$P = -\frac{dE}{dt} \implies \int_0^\tau dt = -\int \frac{dE}{P}.$$

The Larmor formula says that the power radiated by an accelerating charge is given by

$$P = \gamma a^2 \implies \tau = -\int \frac{dE}{\gamma a^2}.$$

The energy of an electron orbiting a nucleus is given by

$$E = \frac{1}{2}mv(r)^2 - \frac{\alpha}{r} \implies dE = \frac{dE}{dr}dr = \Big(mv(r)v'(r) + \frac{\alpha}{r^2}\Big)dr.$$

We now simplify this expression. An electron in uniform circular motion obeys

$$\frac{mv(r)^2}{r} = \frac{\alpha}{r^2} \implies v(r) = \sqrt{\frac{\alpha}{mr}} \implies$$
$$v'(r) = \sqrt{\frac{\alpha}{m}}\frac{-1}{2}r^{-3/2} \implies mv(r)v'(r) = m\sqrt{\frac{\alpha}{m}}r^{-1/2}\sqrt{\frac{\alpha}{m}}\frac{-1}{2}r^{-3/2} = -\frac{\alpha}{2r^2} \implies$$
$$dE = \frac{\alpha}{2r^2}dr \implies \tau = -\int_{r_0}^{0} \frac{\alpha}{2r^2}\frac{1}{\gamma a^2}dr.$$

Again, due to circular motion,

$$a(r) = \frac{\alpha}{mr^2} \implies \frac{1}{\gamma a(r)^2} = \frac{mr^4}{\gamma\alpha^2} \implies$$

$$\tau = -\int_{r_0}^{0} \frac{\alpha}{2r^2} \frac{m^2 r^4}{\gamma \alpha^2} dr = \frac{m^2}{2\gamma\alpha} \int_0^{r_0} r^2 dr \implies$$

$$\boxed{\tau_{\text{atom}} = \frac{m_e^2}{2\gamma\alpha} \frac{r_0^3}{3}.}$$

We now plug in values.

$$\gamma = \frac{e^2}{6\pi\epsilon_0 c^3} \quad , \quad \alpha = \frac{e^2}{4\pi\epsilon_0} \quad , \quad r_0 = 5 \cdot 10^{-11} m \implies$$

$$\boxed{\tau_{\text{hydrogen}} = 1.3 \cdot 10^{-11} s.}$$

This is in manifest contradiction with the stability of matter in the universe.