Exercise We derive the circular orbits of massive and massless particles in the Schwarzchild metric

$$ds^2 = -(1-2GM/r)dt^2 + (1-2GM/r)^{-1}dr^2 + r^2 d\Omega^2$$

(Source: Carroll, 5.4)

If we have a Killing vector K^μ, then $K_\mu \frac{dx^\mu}{d\lambda}$ will be a constant for the free particle moving along the geodesic. Since the metric is static and spherically symmetric (and since we have already fixed motion to be in the $\theta = \pi/2$ plane), we will have two Killing vectors,

$$K^\mu = (\partial_t)^\mu = (1,0,0,0) \quad \text{and}$$

$$R^\mu = (\partial_\phi)^\mu = (0,0,0,1).$$

We can then lower the indices using the Schwartzchild metric and compute the inner product above. We arrive at two conserved quantities

$$K_\mu \frac{dx^\mu}{d\lambda} = -(1-2GM/r)\frac{dt}{d\lambda} \equiv -E$$

$$R_\mu \frac{dx^\mu}{d\lambda} = r^2\sin^2\theta \frac{d\phi}{d\lambda} = r^2\frac{d\phi}{d\lambda} \equiv L,$$

where E will be the energy for the case of the massless particle and the energy per unit mass for the massive particle. L will correspondingly be the conserved angular momentum or angular momentum per unit mass, respectively.

Let $\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for massless particles. Then,

$$g_{\mu\nu} U^\mu U^\nu = -\varepsilon = -(1-2GM/r)\left(\frac{dt}{d\lambda}\right)^2 +$$

$$(1-2GM/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2 \underbrace{\cancel{\left(\frac{d\theta}{d\lambda}\right)^2}}_{\to 0} + r^2\left(\frac{d\phi}{d\lambda}\right)^2 \Rightarrow$$

$$-\varepsilon = -E^2(1-2GM/r)^{-1} + (1-2GM/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + L^2/r^2$$

$$\Leftrightarrow \boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{(1-2GM/r)}{2}\frac{L^2}{r^2} + \varepsilon\frac{(1-2GM/r)}{2} = \frac{E^2}{2}}$$

This defines the effective potential energy

$$\boxed{V_{eff}(r) = \frac{(1-2GM/r)}{2}\frac{L^2}{r^2} + \varepsilon\frac{(1-2GM/r)}{2}}$$

which looks similar to the Newtonian effective gravitational potential, except with a new term $\sim r^{-3}$.

We find the circular orbits for massive particles $(\varepsilon = 1)$:

$$V_{eff}'(r) = 0 = \frac{d}{dr}\left(\frac{L^2}{2r^2} - \frac{GML^2}{r^3} + \frac{1}{2} - \frac{GM}{r}\right)$$

$$= -\frac{L^2}{r^3} + \frac{3GML^2}{r^4} + \frac{GM}{r^2} \implies 0 = 3GML^2 - L^2 r + GMr^2$$

$$\implies r_\pm = \frac{1}{2GM}\left(L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}\right) = \boxed{\frac{L^2}{2GM}\left(1 \pm \sqrt{1 - 12G^2M^2/L^2}\right)}$$

Next, we analyze the stability of these orbits

$$V_{eff}''(r) = \frac{3L^2}{r^4} - \frac{12GML^2}{r^5} - \frac{2GM}{r^3}$$

$$= \frac{1}{r^5}\left(3L^2 r - 12GML^2 - 2GMr^2\right)$$

As the stability will depend on L, we consider the limit $L/GM \gg 1 \rightarrow$

$$r_\pm = \frac{L^2}{2GM}\left(1 \pm \left(1 - 6G^2M^2/L^2\right)\right) \longrightarrow r_+ = \frac{L^2}{GM}\left(1 - 3G^2M^2/L^2\right) \approx$$

$\frac{L^2}{GM}$, $r_- = 3GM$. We now examine the stability of these limiting cases:

$$r_+^5 V''_{eff}(r_+) = 3L^2 \cdot \frac{L^2}{GM} - 12GML^2 - 2GM\frac{L^4}{G^2M^2} =$$

$$\frac{L^4}{GM}\left(1-12\left(\frac{GM}{L}\right)^2\right) \approx \frac{L^4}{GM} > 0.$$

So $\boxed{r_+ \text{ is stable}}$.

We also have

$$r_-^5 V_{eff}(r_-) = 3L^2 r_- - 12GML^2 - 2GMr_-^2 =$$

$$3L^2 \cdot 3GM - 12GML^2 - 2GM \cdot 9G^2M^2 =$$

$$GML^2\left(-3 - 18(GM/L)^2\right) < 0 \rightarrow \boxed{r_- \text{ is unstable}}$$

As we decrease L, the outer stable circular orbit will approach the inner unstable circular orbit. For some L, these two orbits will coincide, and below this value of L, no circular orbits will exist. The value of L for which the inner and outer orbits coincide is the value which kills the radical

$$r_\pm = \frac{L^2}{2GM}\left(1 \pm \sqrt{1-12G^2M^2/L^2}\right) \mapsto r^* = \frac{L^2}{2GM},\ L = \sqrt{12}\ GM \Rightarrow$$

$\boxed{r^* = 6GM}$. This is the radius of the "innermost stable circular orbit" (ISCO).

We now find the circular orbits for massless particles:

$$V_{eff}(r) = \frac{(1-2GM/r)}{2}\frac{L^2}{r^2} = \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \implies$$

$$V_{eff}'(r) = -\frac{L^2}{r^3} + \frac{3GML^2}{r^4} = 0 \implies$$

$$3GML^2 - rL^2 = 0 \implies \boxed{r_c = 3GM},$$

which is the same as the unstable circular orbit for massive particles. We find the stability of this orbit by once again computing the second derivative:

$$V_{eff}''(r) = \frac{3L^2}{r^4} - \frac{12GML^2}{r^5} \implies V_{eff}''(r_c) =$$

$$\frac{3L^2}{(3GM)^4} - \frac{12GML^2}{(3GM)^5} = \frac{L^2}{G^4M^4}\left(\frac{1}{3^3} - \frac{4}{3^4}\right) < 0,$$

so r_c is <u>unstable</u> for massless particles.