A Quick and Dirty Solution to the Catenary Differential Equation

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We present a solution to the differential equation which arises in the catenary problem. One can pose and solve this problem in many ways, but the equation in question is the result of a variational minimization of the energy functional

$$U[y(x)] = \int_{-L/2}^{L/2} y(x)\sqrt{1+y'(x)^2}dx \tag{1}$$

The equation we wish to solve is the following

$$\begin{aligned} 1+y'(x)^2 &= y(x)y''(x) \\ y(-L/2) &= y(L/2) \end{aligned} \tag{2}$$

Given enough time, we could probably guess a solution to this equation in terms of the nice functions we know. However, to demonstrate that it can be solved directly, we elect to take the scenic route. We can start by differentiating both sides of the differential equation to get

$$\begin{aligned} 2y'y'' &= y'y'' + yy''' \\ y'y'' &= yy''' \end{aligned} \tag{3}$$

and we can divide both sides by yy'' to get

$$\frac{y'}{y} = \frac{y'''}{y''} \tag{4}$$

We can now integrate both sides of (4), yielding

$$\begin{aligned} \int_0^x \frac{y'}{y}dx &= \int_0^x \frac{y'''}{y''}dx \\ \ln\left(\frac{y(x)}{y(0)}\right) &= \ln\left(\frac{y''(x)}{y''(0)}\right) \\ \frac{y(x)}{y(0)} &= \frac{y''(x)}{y''(0)} \\ y''(x) &= \frac{y''(0)}{y(0)}y(x) \end{aligned} \tag{5}$$

We have now reduced the problem to a well known linear differential equation. Recall, from your kindergarten differential equations class that (5) is solved by

$$y(x) = A\cosh\left(\sqrt{\frac{y''(0)}{y(0)}}x\right) + B\sinh\left(\sqrt{\frac{y''(0)}{y(0)}}x\right) \tag{6}$$

where A and B are constants. The boundary condition in (2) requires that $B = 0$, and we are left with

$$y(x) = A\cosh\left(\sqrt{\frac{y''(0)}{y(0)}}x\right) \tag{7}$$

Since (7) must also satisfy (2), we must have

$$y(0)y''(0) = 1 \tag{8}$$

And since $\cosh(0) = 1$, we can simplify our solution to

$$y(x) = y(0)\cosh\left(\frac{x}{y(0)}\right) \tag{9}$$

It is important to note that this solution does not admit arbitrary choices for $y(L/2)$ *and* the rope length. However, the minimum energy configuration ought not to depend on how high up the rope is strung. Luckily, we are free to shift $U[y(x)]$ by a constant with impunity, despite its nonlinearity in $y(x)$. Therefore, if $y(x)$ minimizes (1), and the rope length, S, is specified, then shifting $y(x)$ to $y(x) + h$ still minimizes (1), since

$$\begin{aligned} U[y(x)+h] &= \int_{-L/2}^{L/2} (y(x)+h)\sqrt{1+y'(x)^2}dx \\ U[y(x)+h] &= \int_{-L/2}^{L/2} y(x)\sqrt{1+y'(x)^2}dx + h\int_{-L/2}^{L/2} \sqrt{1+y'(x)^2}dx \\ U[y(x)+h] &= \int_{-L/2}^{L/2} y(x)\sqrt{1+y'(x)^2}dx + hS \\ U[y(x)+h] - hS &= \int_{-L/2}^{L/2} y(x)\sqrt{1+y'(x)^2}dx \end{aligned} \tag{10}$$

And we can redefine $U[y(x) + h]$ to be $U[y(x) + h] - hS$. Thus, if L is given, the only parameter we need to specify is the rope length in order to determine $y(x)$.

For some closure, let the rope length be S and assume $S > L$. Then

$$\begin{aligned} y(x) &= y(0)\cosh\left(\frac{x}{y(0)}\right) \\ S &= \int_{-L/2}^{L/2} \sqrt{1+y'(x)^2}dx \\ \implies S &= \int_{-L/2}^{L/2} \cosh\left(\frac{x}{y(0)}\right)dx \\ S &= 2y(0)\sinh\left(\frac{L}{2y(0)}\right) \end{aligned} \tag{11}$$

which, unfortunately, has no inverse in terms of elementary functions. ♣
