

1.1 (2.6) states

$$\sum_{j=1}^3 M_{ij} x_j = k_i$$

$$\rightarrow k_2 \Rightarrow \sum_{j=1}^3 M_{2j} x_j = M_{21} x_1 + M_{22} x_2 + M_{23} x_3 =$$

$$M_{21} x + M_{22} y + M_{23} z =$$

$$6x + 5y + 3z = 7. \checkmark$$

$$k_3 = \sum_{j=1}^3 M_{3j} x_j = M_{31} x_1 + M_{32} x_2 + M_{33} x_3 =$$

$$M_{31} x + M_{32} y + M_{33} z =$$

$$2x - y + 0 \cdot z = 4 \checkmark$$

2.1

We consider the case of two equations in four unknowns:

$$k_1 = \sum_{j=1}^4 M_{1j} x_j = M_{11} x_1 + M_{12} x_2 + M_{13} x_3 + M_{14} x_4$$

$$k_2 = \sum_{j=1}^4 M_{2j} x_j = M_{21} x_1 + M_{22} x_2 + M_{23} x_3 + M_{24} x_4.$$

Next, we consider the case of four equations in two unknowns:

$$k_1 = \sum_{j=1}^2 M_{1j} X_j = M_{11} X_1 + M_{12} X_2$$

$$k_2 = \sum_{j=1}^2 M_{2j} X_j = M_{21} X_1 + M_{22} X_2$$

$$k_3 = \sum_{j=1}^2 M_{3j} X_j = M_{31} X_1 + M_{32} X_2$$

$$k_4 = \sum_{j=1}^2 M_{4j} X_j = M_{41} X_1 + M_{42} X_2.$$

3.1

$$\begin{aligned} x - 2y + 13 &= 0 \\ y - 4x &= 7 \end{aligned} \rightarrow \begin{pmatrix} 1 & -2 & -13 \\ -4 & 1 & 17 \end{pmatrix} \xrightarrow{+4I}$$

$$\begin{pmatrix} 1 & -2 & -13 \\ 0 & -7 & -35 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{+2II}$$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{pmatrix} \rightarrow \boxed{x = -3, y = 5}$$

4.

$$\begin{array}{l} 2x + y - z = 2 \\ 4x + y - 2z = 3 \end{array} \rightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 4 & 1 & -2 & 3 \end{pmatrix} \xrightarrow{-2I} \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & -1 \end{pmatrix} \xrightarrow{+II}$$

$$\rightarrow \begin{pmatrix} 2 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\boxed{\begin{array}{l} x = \frac{1}{2}(1+z) \\ y = 1 \end{array}}$$

This system has infinitely many solutions.

$$\begin{array}{l} 5. \ 2x + y - z = 2 \\ \quad 4x + y - 2z = 3 \end{array} \rightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{-2I}$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{This system is inconsistent.$$

$$\begin{array}{l} 6. \ x + y - z = 1 \\ \quad 3x + 2y - 2z = 3 \end{array} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{-3I}$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{+II} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \rightarrow \boxed{x=1, y=z}$$

7.1

$$\begin{array}{l} 2x+3y=1 \\ x+2y=2 \\ x+3y=5 \end{array} \rightarrow \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix} \begin{array}{l} -I \\ -I \\ -2I \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -2II \\ \\ \end{array} \rightarrow \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\boxed{\begin{array}{l} x = -4 \\ y = 3 \end{array}}$$

8.1

$$\begin{array}{l} -x+y-z=4 \\ x-y+2z=3 \\ 2x-2y+4z=6 \end{array} \rightarrow \begin{pmatrix} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 2 & -2 & 4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & -4 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ -I \\ \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & -11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\boxed{\begin{array}{l} x = y - 11 \\ z = 7 \end{array}}$$

9.1

$$\begin{array}{l} x - y + 2z = 5 \\ 2x + 3y - z = 4 \\ 2x - 2y + 4z = 6 \end{array} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 5 \\ 2 & 3 & -1 & 4 \\ 2 & -2 & 4 & 6 \end{pmatrix} \begin{array}{l} \\ -2I \\ -2I \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 5 & -5 & -6 \\ 0 & 0 & 0 & -4 \end{pmatrix}. \text{ This system is } \underline{\text{inconsistent}}.$$

10.1

$$\begin{array}{l} x + 2y - z = 1 \\ 2x + 3y - 2z = -1 \\ 3x + 4y - 3z = -4 \end{array} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & -2 & -1 \\ 3 & 4 & -3 & -4 \end{pmatrix} \begin{array}{l} \\ -2I \\ -3I \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -2 & 0 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \text{ This system is } \underline{\text{inconsistent}}.$$

11.1

$$\begin{array}{r} x - 2y = 4 \\ 5x + z = 7 \\ x + 2y - z = 3 \end{array} \rightarrow \begin{pmatrix} 1 & -2 & 0 & 4 \\ 5 & 0 & 1 & 7 \\ 1 & 2 & -1 & 3 \end{pmatrix} \begin{array}{l} -5I \\ -I \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 10 & 1 & -13 \\ 0 & 4 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & -1/4 & -1/4 \\ 0 & 1 & 1/10 & -13/10 \end{pmatrix} \begin{array}{l} +2II \\ -II \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & -1/2 & 7/2 \\ 0 & 1 & -1/4 & -1/4 \\ 0 & 0 & 7/20 & -21/20 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 & 7/2 \\ 0 & 1 & -1/4 & -1/4 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{array}{l} +\frac{1}{2}III \\ +\frac{1}{4}III \end{array} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{pmatrix} \rightarrow \boxed{x=2, y=-1, z=-3}$$

12.1

$$\begin{array}{r} 2x + 5y + z = 2 \\ x + y + 2z = 1 \\ x + 5z = 3 \end{array} \rightarrow \begin{pmatrix} 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 5 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 2 & 5 & 1 & 2 \end{pmatrix} \begin{array}{l} -I \\ -2I \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 5 & -9 & -4 \end{pmatrix} \begin{array}{l} -5II \\ -5II \end{array} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{l} -5III \\ +3III \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad \boxed{x = -2, y = 1, z = 1}$$

13.1

$$\begin{aligned} 4x + 6y - 12z &= 7 \\ 5x - 2y + 4z &= -15 \\ 3x + 4y - 8z &= 4 \end{aligned} \rightarrow \begin{pmatrix} 4 & 6 & -12 & 7 \\ 5 & -2 & 4 & -15 \\ 3 & 4 & -8 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{3}{2} & -3 & \frac{7}{4} \\ 5 & -2 & 4 & -15 \\ 3 & 4 & -8 & 4 \end{pmatrix} \begin{matrix} \\ -5I \\ -3I \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & \frac{3}{2} & -3 & \frac{7}{4} \\ 0 & -\frac{1}{2} & 19 & -\frac{15}{4} \\ 0 & -\frac{1}{2} & 1 & -\frac{5}{4} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{3}{2} & -3 & \frac{7}{4} \\ 0 & -\frac{1}{2} & 1 & -\frac{5}{4} \\ 0 & -\frac{1}{2} & 19 & -\frac{15}{4} \end{pmatrix} \begin{matrix} +3II \\ \rightarrow \\ -19II \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & -\frac{1}{2} & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\boxed{\begin{aligned} x &= -2 \\ y &= 2z + \frac{5}{2} \end{aligned}}$$

This system has infinitely many solutions.

14.1

$$\begin{aligned} 2x + 3y - z &= -2 \\ x + 2y - z &= 4 \\ 4x + 7y - 3z &= 11 \end{aligned} \rightarrow \begin{pmatrix} 2 & 3 & -1 & -2 \\ 1 & 2 & -1 & 4 \\ 4 & 7 & -3 & 11 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & -1 & -2 \\ 4 & 7 & -3 & 11 \end{pmatrix} \begin{matrix} -2I \\ -4I \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -10 \\ 0 & -1 & 1 & -5 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & 5 \end{pmatrix}. \quad \text{This system is inconsistent.$$

15.

To determine the rank, we row-reduce these matrices.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 5 \end{pmatrix} \begin{matrix} -2I \\ -3I \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{rank} = \boxed{2}$$

16.

$$\begin{pmatrix} 2 & -3 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3/2 & 5/2 & 3/2 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \end{pmatrix} \begin{matrix} -4I \\ -3I \end{matrix} \rightarrow$$



$$\begin{pmatrix} 1 & -3/2 & 5/2 & 3/2 \\ 0 & 5 & -1 & -5 \\ 0 & 5/2 & -1/2 & -1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3/2 & 5/2 & 3/2 \\ 0 & 5 & -1 & -5 \\ 0 & 5 & -1 & -1/2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -3/2 & 5/2 & 3/2 \\ 0 & 1 & -1/5 & -1 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \xrightarrow{+3/2 \text{ II}} \begin{pmatrix} 1 & 0 & -1/5 & 0 \\ 0 & 1 & -1/5 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\text{rank} = \boxed{3}$$

17.

$$\begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 1 & 10 & 7 \\ 4 & 2 & 14 & 10 \\ 2 & 0 & 6 & 4 \end{pmatrix} \begin{matrix} \\ -3I \\ -4I \\ -2I \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 4 & 3 \\ 0 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\text{rank} = \boxed{2}$$

18.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 2 & 2 & 5 & 3 \\ 2 & 4 & 8 & 6 \end{pmatrix} \begin{matrix} \\ +I \\ -2I \\ -2I \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 4 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}II} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = \boxed{3}$$